Problem Overview:

Rainwater flows into a drainpipe at a rate given by $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t measured in hours, and $0 \le t \le 8$. Water drains out of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \le t \le 8$. At time $t = 0$ there are 30 cubic feet of water in the pipe.

Part a:

Students were asked to calculate the number of cubic feet of rainwater that flow into the pipe for $0 \le t \le 8$.

Part b:

Students were asked to determine whether the amount of water in the drainpipe is increasing or decreasing at time $t = 3$ and give a reason for the answer.

Part c:

Students were asked to determine the time t for $0 \le t \le 8$ when the amount of water in the drainpipe is at a minimum and give a reason for the answer.

Part d:

Water continues to flow into and out of the pipe at the given rates for $t > 8$. The pipe can only hold 50 cubic feet of water before overflowing. Students were asked to write, but not solve, an equation involving one or more integrals to find the time w when the pipe begins to overflow.

Part a:

This is an accumulation of water that can be found using an integral of the rate of change in the water flowing into the pipe. A definite integral $\int_0^8 R(t)\,dt$ is sufficient because the rate given is positive on the given interval and describes water going into the pipe. An indefinite integral was sufficient to earn the first point. The limits on the integral were part of the second point, awarded in the presence of that definite integral and the correct answer.

Part b:

Students needed to compare $R(3)$ to $D(3)$ to answer this question. The first of the two points awarded in this part of the question was for convincing evidence that both of these values were being considered, with the $R(3)$ and $D(3)$ as such, or with numerical values appropriate for these functions at $t = 3$. The answer for the second point had to be "decreasing" and include some reference to the relationship between $R(3)$ and $D(3)$ such as $R(3) < D(3)$. However, an answer with support such as $R < D$ and no other numerical values (for example, not even the 3 was shown) could only get the second point, not the first. Completely leaving out numerical values and the $t = 3$ was scored as though not enough work was shown by the student and earned at most one of the two points.

Part c:

Since this problem specifies a closed interval, students should have been looking for an absolute minimum on the interval $0 \le t \le 8$. Considering, and even finding, the value of t for which $R(t) - D(t) = 0$ was worth one point. "Consideration" of this could be shown by a sketch of the graphs of R and D intersecting or a graph of $R - D$ intersecting the x-axis or using equations such as $R = D$. The value of $t = 3.271$ or 3.272 earned the second point. The absolute minimum justification should have shown values of the amount of water in the pipe at the endpoints, $t = 0$ and $t = 8$. Students talking about a change in sign from negative to positive for $R - D$, and not the endpoints, did not often earn the third point. This "relative minimum" argument had to correctly refer to signs over intervals that included values of $t = 0$ and $t = 8$ in their explanations. A few students continued to "talk" in their justifications, making it unclear that 3.271 or 3.272 was the correct (only) value of t and thus lost the last two points.

Part d:

The first of two points awarded in this part of the problem was for a definite integral. This had to be either in an equation or an expression with BOTH either 0 or 8 and a variable as limits AND $R - D$ or $D - R$ as an integrand. Despite a missing differential in an expression, this point was awarded unless of the ambiguous form $\int_0^x (R - D) + 30$. An integral such as $\int_0^x (R - D)$ earned the first point despite not using w.

The second point was earned for an equation that would lead to a solution for w. This included the use of values related to the amount of water such as $\int_8^x (R - D)dt + 48.544 > 50$, an unusual way to use the amount of water at $t = 8$ in an equation. Students could earn this point provided they had earned the first point or had only a minor sign error in the setup of the integral.

Observations and recommendations for teachers:

(1) In order to show work in part (a) students needed to start with a definite integral of the rate of change of the water going into the pipe. The setup of a good integrand is worth a point, and that is not unusual on the AP Calculus exam. This can be more involved than in this problem, for example, in cases where the rate of change changes sign. Working with the difference between net change and total change (such as displacement $\int_{t_1}^{t_2} v(t)\,dt$ vs. total distance $\int_{t_1}^{t_2} |v(t)|\,dt$) should be practiced in a variety of contexts. Past AP Exam questions such as 2011 AB1 or 2012 AB6 provide examples.

(2) Part (b) is an example of a problem where considering, in writing, an appropriate relationship involving the necessary derivatives earns a point. However, some values need to be shown in order to get full credit for a justification of the correct answer, either increasing or decreasing. See 2013 AB1 part (c) for an example.

(3) Part (d) is an example of not just locating an extreme value, but finding an absolute extreme value because we are looking at a closed interval. See 2013 AB1 part (d) for another example. If the endpoint and critical point(s) values are shown, the justification is complete without the difficulty of referring to a change in signs over appropriate intervals.

(4) Students should be required to provide the differential in setting up an integral to avoid any ambiguity. A problem such as part (d) is common on the AP Calculus Exam. See 2013 AB2 part (c) for another example.