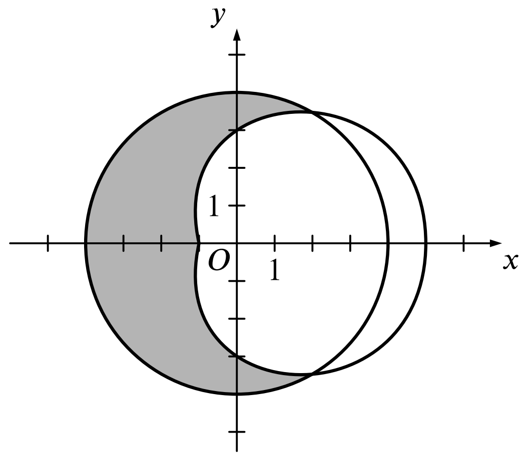
Problem Overview

The student is given the graph of the circle $r = 4$ and the cardioid $r = 3 + 2\cos\theta$. The student is told that they intersect when $\theta = \dfrac{\pi}{3}$ and when $\theta = \dfrac{5\pi}{3}$.



Part a

The student is asked write an integral expression that evaluates to the area of the shaded region.

Part b

The student is asked to find the slope of the tangent to $r = 2 + 3\cos\theta$ at the point where $\theta = \dfrac{\pi}{2}$.

Part c

The student is told that a particle is moving along the portion of the curve $r = 2 + 3\cos\theta$ that is in the first quadrant such that the distance from the origin to the particle is increasing at 3 units per second. The student is asked to find the rate at which the angle θ changes with respect to time at $\theta = \pi/3$. The student is asked for the units of this rate as well.

Comments on Student Responses and Scoring Guidelines

Part a

The student earns one point by using the correct limits and the constant $1/2$. As the intersection points were stated in the problem, this was an easy point to earn. Students who noted the symmetry of the problem integrated from $\theta = \pi/3$ to $\theta = \pi$ and doubled the integral; this also earned the point. Students earned two points by having the correct integrand $4^2 - (2 + 3\cos\theta)^2$. Equivalent expressions were accepted, but if the integrand was not a difference of squares students did not earn the points.

Part b

To earn the first point, the student had to have at least one correct derviative of $x = (2 + 3\cos\theta)\cos\theta$ or $y = (2 + 3\cos\theta)\sin\theta$. To earn the second point, the student had to assemble their (possibly incorrect) derivatives of x and y correctly to form dy/dx. Finally, to earn the third point, the student had to evaluate dy/dx correctly at $\theta = \pi/2$.

Part c

The student earned one point for demonstrating the chain rule. This was mostly indicated by the presence of $d\theta/dt$ when differentiating $r = 2 + 3\cos\theta$. The second point was earned in isolating $d\theta/dt$ correctly – in other words, for using correct algebra. The third point was for evaluating $d\theta/dt$ correctly at $\theta = \pi/3$. Any algebra or arithmetic errors did not earn the second point. Any errors in trigonometric evaluation did not earn the third point. So a student who made an algebra error and a trig error did not earn the last two points on this part of the problem.

Observations and Recommendations for Teachers

(1) Students should avoid unnecessary work. In part (a), even though the intersection points were given, some students still solved for them, and most did so incorrectly. Some students expanded the squares and simplified. If a student did this algebra incorrectly and declared the incorrect work as their answer, the points were not earned. Some students did not read the problem and computed the area of the shaded region; again, most who did this were incorrect.

(2) Students should avoid unnecessary work. In part (b), students did not need to evaluate the trigonometric functions at $\theta = \pi/2$. An answer such as

$$\frac{dy}{dx} = \frac{-3\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + \left(2 + 3\cos\left(\frac{\pi}{2}\right)\right)\left(\cos\left(\frac{\pi}{2}\right)\right)}{-3\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \left(2 + 3\cos\left(\frac{\pi}{2}\right)\right)\left(-\sin\left(\frac{\pi}{2}\right)\right)}$$

was perfectly wonderful, and could have earned all three points by itself. Many students did not know their trigonometric values, and a surprising number thought they were using $\pi/4$ instead of $\pi/2$. (Indeed, there were more than a few students who converted the equation to rectangular and found the derivative implicitly. While this method is acceptable, less than a few carried this through to a correct numerical value.)

(3) Students should avoid reciting formulas. Formulas (or “recipes”) are not acceptable on the AP Exam.

Students who thought they would get at least one point by only writing

$$\frac{r' \sin\theta + r\cos\theta}{r' \cos\theta - r\sin\theta}$$

did not, since there was no evidence that they ever got into the stated problem. Nor did a student who wrote the formula for the derivative and then magically wrote the correct answer $2/3$ without any evidence that they took derivatives.

(4) A general comment: Students should think carefully before boxing or circling their answers. A boxed or circled answer is treated as the answer the student wants declared as their answer. The correct answer could be on their paper with the correct work, but in the presence of a boxed or circled incorrect answer, the student will not earn the answer point.