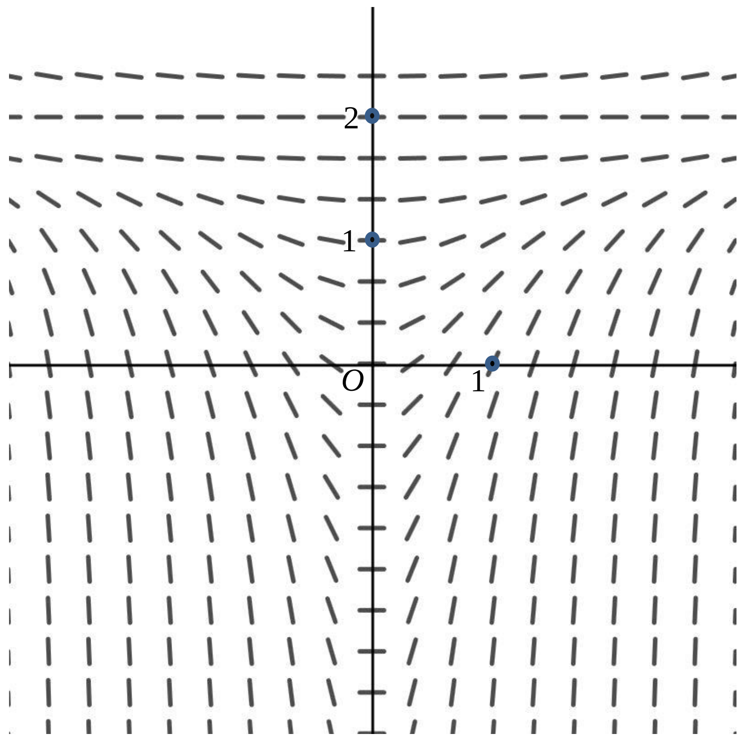
Problem Overview:

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

Part a:

A slope field for the differential equation, similar to the one given below, was shown. Students were asked to sketch the solution curve that passes through the point (0, 2) and sketch the solution curve that passes through the point (1, 0).



Part b:

Given that $y = f(x)$ is the particular solution to the differential equation with initial condition $f(1) = 0$, students were asked to write an equation of the line tangent to the graph of $y = f(x)$ at $x = 1$ and use this equation to approximate $f(0.7)$.

Part c:

Students were asked to find the particular solution $y = f(x)$ with initial condition $f(1) = 0$.

Comments on student responses and scoring guidelines:

Part a:

One point was awarded for each curve sketched correctly. Curves had to contain the point, follow the slope field, and extend to the boundary of the given slope field. The curve through the point (0, 2) had to be a horizontal line. The curve through the point (1, 0) had to be relatively smooth, have a horizontal tangent at the y-axis, and not cross the line $y = 2$. Blatant concavity errors in the non-linear sketch would lose the point.

Part b:

After the first appearance of a correct tangent line as in $y = \frac{4}{3}(x-1)$ or $y = \frac{4}{3}x - \frac{4}{3}$ or an equivalent, the student earned one point, and any subsequent errors came off the second point in this part of the problem. The approximation could be shown in many equivalent forms such as -0.4 or $-\frac{6}{15}$ or $\frac{4}{3}(-0.3)$. Work had to be shown using the slope. A bald answer of -0.4 earned zero points. Work such as $y = \frac{4}{3}(0.7-1)$ earned the second point but not the first, which required that an equation of the tangent line be shown. Students were not penalized for writing $f(0.7) = \frac{4}{3}(0.7-1)$ rather than correctly writing $f(0.7) \approx \frac{4}{3}(0.7-1)$.

Part c:

The first of five points was for correctly separating the variables as in $\frac{dy}{(y-2)^2} = \frac{1}{3}x\,dx$ or $\int \frac{1}{(y-2)^2} = \int \frac{x}{3}$. Some students launched into antiderivatives presenting something like $\frac{-1}{y-2} = \frac{1}{6}x^2$ which earned the separation point and both antiderivative points. Some copy errors such as $\frac{dy}{(y+2)^2} = \frac{1}{3}x\,dx$ did not earn the first point but were eligible for both antiderivative points. If x was missing from the right side, the antiderivatives on both left and right sides had to be calculated correctly in order for the student to earn only one of the antiderivative points. In order to earn the fourth point, students had to include $+C$ properly and show appropriate use of the initial condition. Students were eligible for this point provided at least one antiderivative point had been earned. Students were not eligible for this fourth point if a previous error created a situation in which C was not a real number. Any arithmetic errors in solving for C came off the fifth point which was for the answer, the expression for y. The domain did not need to be stated in order to earn the answer point. Using a definite integral, the initial condition, and dummy variables, a complete solution can also be found by correctly computing $\int_0^y \frac{d\alpha}{(\alpha-2)^2} = \int_1^x \frac{\beta}{3}\,d\beta$ and solving for y.

Observations and recommendations for teachers:

(1) A slope field indicates a family of solutions to this form of a differential equation. Some students had trouble following the indicated slopes needed to sketch the non-linear solution passing through (1, 0). This needs to be practiced with instructions to extend the curve to the boundaries of the given slope field. Arrows are not sufficient to extend a non-linear solution sketch to the boundaries because work had to be shown indicating an accurate tracing through the given point *and* following the indicated slopes to the boundaries.

(2) Some students had difficulty determining a horizontal line was needed for one of the curves in part a. It is possible that the word "curve" is not used enough in teaching because in this and other situations, a curve may refer to a straight line.

(3) The given differential equation *is* the slope of lines tangent to the graph of a solution $y = f(x)$. Numerical slope at the initial condition point can be calculated by substituting the initial condition into the expression for $\frac{dy}{dx}$.

(4) Often, an equation of a tangent line is best written using point-slope form as in $y - y_1 = m\left(x - x_1\right)$. Students should carefully answer the question asked which was for an equation of a tangent line. An approximation to a value of the function is found by simply substituting the given value of x. But be sure to solve for y because the approximation is a "*y*-value," *not* an equation such as $y + \frac{4}{3} = \frac{4}{3}(0.7)$. Since this is an *approximation*, students using function notation should have been trained to write $f(0.7) \approx \frac{4}{3}(0.7 - 1)$ rather than using an equal sign.

(5) Although differentials have mathematical meanings that go beyond the intent of this problem, the first step in separating variables "looks like" multiplying both sides of the equation by dx. Once this is done, simple algebraic operations can rearrange the equation so that all like variables are separated into the two different sides of the equation. With the differentials there, as they should be, this now begs for integration. Just one instance of +*C* is needed. Combining the two possible +*C*'s into one +*C* on one side of the equation is a legitimate way to proceed through the calculations. Both the correct instance of this +*C* and the substitution of the initial condition values for x and y have been needed on the test for students to earn a point after calculating antiderivatives. Solving for y includes calculating *C* and some algebra. Sometimes, this can involve as much work as was done in earning the first four points. If time is an issue for a student taking the test, this last step can be postponed until other, easier, portions of the exam have been addressed.

(6) Teachers should be aware that the fourth point in part c requires both the correct presence of a +*C* and using the initial condition. For many years, a separate point was awarded for each of the +*C* and using the initial condition. When using past AP exam questions such as 2005AB6 part c or 2008AB5 part b for practice, teachers need to remind students that the third and fourth points awarded in 2005 and 2008 are now combined into only one point in the scoring guidelines.