2017 BC-6 Dennis Wilson, Landmark Christian School

Problem Overview:

The problem involved a function f for which derivatives at $x = 0$ were defined recursively as shown to the right. The students were told that the Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

Part a:

Students were asked to show that the first four terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ and to write the general term for the Maclaurin series.

Part b:

Students were asked to determine if the Maclaurin series for f at $x = 1$ converges absolutely, converges conditionally, or diverges. The student was required to give reason for the answer.

Part c:

Students were asked to write the first four nonzero terms and the general term of the Maclaurin series for the function $g(x) = \int_0^x f(t)\,dt$.

Part d:

Students were asked to use Alternating Series Error Bound to show $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$ where $P_n\left(\frac{1}{2}\right)$ represents the nth-degree Taylor polynomial for g evaluated at $x = \frac{1}{2}$.

General scoring guidelines for the problems:

First, a general note about scoring this question. The terms in either part (a) or part (c) may be listed or added. If higher degree terms were given by students, the reader was instructed to ignore them. Readers were to ignore the tail of the series $(+\cdots)$ as well as any use of sigma notation with or without indices.

Part a: (3 points)

The first point in the problem was awarded for numerical values of the derivatives. These could be listed or they could be combined with factorials. In either case, the recursive form must be resolved. The derivatives could also be imbedded in the verification.

The second point was given for the verification of the first four non-zero terms. All three parts of the terms (derivative, factorial, and power of x) must exist in a form which was not completely simplified.

Students could arrive at both the derivatives and verification without using the recursive formula. Students could derive a general form of the derivative $f^{(n)}(0) = (-1)^n(n-1)!$ or they could recognize that $f(x) = \ln(1+x)$. The student could derive the derivatives and verification from these forms.

The third point was for the general term. The general term for the series must be given in this part for the student to earn this point. The parity (even/odd) of the power of -1 and x must be different. An example of the minimal expression earning all three points would be:

$$x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots$$

Common Mistakes: Students who struggled with this part generally did so because they failed to show the numeric values of the derivatives. They failed to handle the recursive nature of the derivative. Many students did not attempt to give the general term after failing in the verification of the first four.

<u>Part b:</u> (2 points)

First, the convergence of the series must be considered at $x = 1$, or the student receives no points.

This part has three pieces and a student must handle all three pieces sufficiently to earn both points. The three parts are the convergence of $\frac{\sum(-1)^{n+1}}{n}$, the divergence of $\sum\frac{1}{n}$, and the conclusion of conditional convergence. Students were required to show work in their analysis.

Sufficient examples for the convergence of $\frac{\sum(-1)^{n+1}}{n}$:	Sufficient examples for the divergence of $\sum\frac{1}{n}$:
$\lim_{n\to\infty}\frac{1}{n} = 0$, $\frac{1}{n} > \frac{1}{n+1}$, therefore $\frac{\sum(-1)^{n+1}}{n}$ converges.	$\sum\frac{1}{n}$ diverges
$\frac{\sum(-1)^{n+1}}{n}$ converges by A.S.T.	Harmonic series diverges
	Harmonic series
$\frac{\sum(-1)^{n+1}}{n}$ is the alternating harmonic series. (Note: it was not necessary using this example to state convergence.	$\sum\frac{1}{n}$diverges by p-series or integral test.

Once sufficient examples were given for both as shown above, the student would earn the second point by concluding that the series is conditionally convergent.

The student could earn one of the two points if he or she gave sufficient examples but came to an incorrect conclusion about conditional convergence. The student could also earn one of the two points by being correct with one of the sufficient examples and reaching a consistent conclusion. A student would only earn one of two points by simply stating that the alternating harmonic series converges.

The reader could read with an incorrect series imported from part (a), as long as the series was conditionally convergent. For example, $\sum \frac{(-1)^{n+1}}{n!}$ is absolutely convergent and therefore could not be imported.

Common Mistakes: Most students who did not receive both points on this part failed in stating the divergence of the harmonic series. It seemed many did not recognize the concept of conditional convergence despite being added as an objective for this exam. Some students argued that the series diverged at $x = 1$ because the stem of the problem stated the series converged for $|x| < 1$.

<u>Part c:</u> (3 points)

The first point came for any two correct terms. The second point was earned for the other two correct terms. If a student presents all four terms with incorrect signs, he or she would earn one of the two points. The reader did not read terms of higher degree if the first four were presented correctly. The third point came for the general term. An incorrect series could be imported from part (a) if it was a consistent alternating series with at least a quadratic denominator.

Common Mistakes: The worst mistake is that many students did not attempt to find the first four terms despite the fact that all four were simple antiderivatives of the series given in part (a). The other common mistake is that students struggled with the parity of the powers in the general terms. Many added one to the power of -1 as well as to x.

<u>Part d:</u> (1 points)

A student must indicate two things to earn the final point on the BC exam. First, the student must present the answer in terms of an error. In other words, the word error needed to be written or the mathematical expression for it. Second, a numeric value less than $\frac{1}{500}$ must be connected in some way to the first neglected term. There must be an explicit and supported reference to $\frac{1}{640}$ but not to $\frac{1}{500}$. Students who used Lagrange Error Bound received no point. An example of a minimal answer would be:

$$Error < \frac{\left(\frac{1}{2}\right)^5}{20}$$

Observations and recommendations for teachers:

1. Students should be encouraged to read through every question and part on the exam. Two very easy points unrelated to series were available on part (c) and yet many students did not attempt question 6 at all, much less part (c).

2. Students should be encouraged to "Answer the question that is asked." On part (d), students did correct calculus but failed to frame their computations in terms of the word error. By simply restating what is being asked, students can insure against missing points for lack of communication. Communication was emphasized in much of the scoring due to its emphasized importance as a mathematical practice of AP Calculus.

3. Students should learn why the harmonic series diverges and the alternating harmonic series converges. Once this is learned though, students should learn that the statements themselves serve as justification for the AP exam.

4. Students need practice in differentiation, integration, and other forms of manipulations of the general term for a series. Students should learn to test the parity of powers of general terms for alternating series using the first four terms.