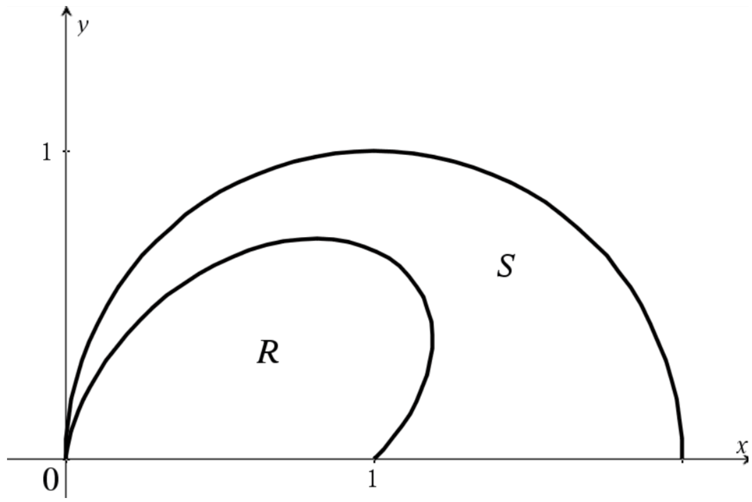
2017 BC-2 Dennis Wilson, Landmark Christian School

**<u>Problem Overview</u>:**

The problem involved two polar curves defined by the functions $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta < \frac{\pi}{2}$. A graph of two regions, $R$ and $S$, bounded by the curves was given. Region $R$ was the region bounded by the graph of $f(\theta)$ and the $x$-axis. Region $S$ was bounded by the graph of $f(\theta)$, the graph of $g(\theta)$, and the $x$-axis.

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**Part a:**

Students were asked to find the area of the region $R$.

**Part b:**

Students were asked to write an equation involving one or more integrals whose solutions would yield the value of $k$, the angle splitting region $S$ into two parts.

**Part c:**

Students were asked to find an expression for $w(\theta)$, the distance between the points $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Students were then asked to find, $w_A$, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

**Part d:**

Students were asked to find the value of $\theta$ for which $w(\theta) = w_A$. Students were asked to determine if $w(\theta)$ was increasing or decreasing at the value of $\theta$ and to give a reason for their answer.

**General scoring guidelines for the problems:**

**Part a: (2 points)**

The first point in this part was awarded for the integral. Two correct forms of the integral were mainly seen:

$$\int_0^{\frac{\pi}{2}} \left(f(\theta)\right)^2 d\theta = \int_0^{\frac{\pi}{2}} (1 + \sin\theta\cos 2\theta)^2\, d\theta$$

Any nonzero constant multiples of this integral earned the first point. The constant multiple did not need to be $\frac{1}{2}$. The limits of integration must be correct.

An indefinite integral could not earn the first point. Students who chose to write $r^2$ instead of $\left(f(\theta)\right)^2$ also could not earn this point. In both of these cases, students could still earn the second point with a correct answer.

| | |
|---|---|
| $\int \left(f(\theta)\right)^2 d\theta$ | 0-? |
| $\int_0^{\frac{\pi}{2}} r^2\, d\theta$ | 0-? |

In the case of missing or incorrect parentheses, a student could still earn both points with a correct answer present. Without a correct answer, the student earned zero points for this part.

The second point for this part came for the answer. No bald answers were accepted. An integral as described before was necessary. This point was awarded for the exact value of $\frac{15\pi-16}{48}$ or for the decimal approximation of $.648$. In the case of a decimal presentation error, the student lost this point but was inoculated against any future deductions for decimal presentation on this problem.

Some students, more than past years, worked through this problem with their calculator in degree mode. (It is believed this is because the Physics exam occurred the day before the Calculus exam.) The first answer given with degree mode lost this answer point. Subsequent answers from degree mode were inoculated against deduction.

Common Mistakes: Overall, students either earned these points or missed it completely. Many students stated that they had never covered polar curves. Some students used $g(\theta)$ for their integral while others multiplied their integral by a constant of $\pi$ instead of $\frac{1}{2}$. Others had the upper limit of integration as 1 or 2 instead of $\frac{\pi}{2}$.

**Part b:** **(2 points)**

Students earned the first point of this part by writing an integral expression for one of the two regions.

$\int_0^k \left( \left(g(\theta)\right)^2 - \left(f(\theta)\right)^2 \right) d\theta$ 1-?

$\int_k^{\pi/2} \left( \left(g(\theta)\right)^2 - \left(f(\theta)\right)^2 \right) d\theta$ 1-?

An integral without one limit of $k$ did not earn this point. The student could miswrite $k$ as $x$. Any nonzero constant multiple of these integrands would earn the first point. Students were eligible for the first point with a parenthesis error but not the second.

The second point was earned with a correct version of the equation. Students could evaluate the numerical approximation for area of the region $S$ and use it in their equation.

$\int_0^k \left( \left(g(\theta)\right)^2 - \left(f(\theta)\right)^2 \right) d\theta = .922$ 1-1

Common Mistakes: The most common mistake was giving the integral as a square of differences instead of a difference of squares. Another common error made by students was misinterpreting the region $S$, using only the function $g(\theta)$ instead of $f(\theta)$ as well.

**Part c:** **(3 points)**

A correct, explicitly stated expression for $w(\theta)$ must be given to earn the first point. While this point was most easily earned with $w(\theta) = g(\theta) - f(\theta)$, student gave a wide variety of responses which earned credit.

$$w(\theta) = \sqrt{\left(g(\theta) - f(\theta)\right)^2 + (\theta - \theta)^2}$$

$$w(\theta) = |f(\theta) - g(\theta)|$$

Some students converted everything to rectangular coordinates and still earned points.

$$w(\theta) = \sqrt{(g(\theta)\cos\theta - f(\theta)\cos\theta)^2 + (g(\theta)\sin\theta - f(\theta)\sin\theta)^2}$$

If students did not explicitly state $w(\theta)$, using it first in an integral, then they did not earn the first point.

The second point for this part was awarded for the integral. If students did not explicitly state $w(\theta)$, they could still earn this point with the correct integrand. If the students stated $w(\theta)$ incorrectly, they could earn this point by using their expression or $w(\theta)$ in the integral. They could not earn the third point though. Readers were not allowed to read with students on incorrect expressions of $w(\theta)$.

The third point was awarded for the correct answer. No bald answers were accepted. This point was only earned with the exact value of $\frac{14-3\pi}{3\pi}$ or the decimal approximation $.485$.

Common Mistakes: Students reversed the order of the difference, giving $w(\theta) = f(\theta) - g(\theta)$. Many students made algebraic mistakes as they attempted to use the distance formula in either polar or cartesian coordinates.

**<u>Part d:</u> (2 points)**

There were two points awarded for this part. The first point was for solving the equation $w(\theta) = w_A$. The point was awarded for the approximate value of $.518$ (rounded) or $.517$(truncated).

The second point was awarded for correctly concluding that $w(\theta)$ is decreasing with an appropriate reason. The appropriate reason was to state in some way that $w'(.518) < 0$. It could come as easily as a sentence such as "decreasing because $w'(.518)$ is negative." Student were not required to state a numerical value for $\theta$, but if they did, the value must be given on the interval $-.6 < \theta < -.5$. It was not enough to state the value of $w'$;, students were required to explicitly state that the value was negative in some way.

Readers were allowed to read with an incorrect value of $w_A$ imported from part (c) under three conditions. First, the student must have correctly stated $w(\theta)$ in part (c). Second, the incorrect value of $w_A$ must be in the interval $[0,1]$. Third, the value of $\theta$ for which students solve must be on the interval $\left[0, \frac{\pi}{2}\right]$.

Common Mistakes: Most students who responded correctly to part $(c)$ were able to earn full points on this part. Some students missed points for not explicitly stating $w'(\theta)$ as negative.

**<u>Observations and recommendations for teachers:</u>**

1. Students should be encouraged to determine if their answers make sense in the context of the problem. Values in all parts of this problem were given by students which could not be possible. Students should understand the procedures they use, but they should also understand what these procedures produce. Students were asked for a value $k$ in part (b) on the interval $\left[0, \frac{\pi}{2}\right]$ and many gave values that were well outside this range. Students were asked to give an average value of a distance between curves. This distance was always greater than one but less than two. Yet students produced answers that were well outside that range. A useful exercise in reviewing this problem might be to discuss possible values for the answers before proceeding with a solution.

2. Students should be able to distinguish between polar curves given their equations and graphs. There is not enough time for students to graph polar curves on their calculators during the examination, especially if the graphs are already given. Students could easily distinguish between the curves just by substituting $\theta = 0$ into the two equations. Unfortunately, many students chose the wrong function in finding the area of region $R$.

3. Students should be encouraged to read the description of the regions in the text of the problem and not just decide based on their interpretation of the picture given.

4. Students should be encouraged to read the entire problem as many students were able to correctly work the last two parts without any knowledge of polar curves.

5. Teachers should review free response questions from previous years. The distance between two polar curves was asked on a question on the 2014 exam. A review of this problem would have saved many students from the algebraic difficulties of using the distance formula in either cartesian or polar form.