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Problem Overview:

A boiled potato is removed from cooking and left to cool at time $t = 0$. At $t = 0$ the internal temperature of the potato is 91° Celsius ($^\circ$C). At all times $t > 0$, this temperature is greater than 27°. At times t in minutes, the internal temperature of the potato can be modeled by a function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$ where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

Part a:

Students were asked to write an equation of the line tangent to the graph of H at the point where $t = 0$ and use this equation to approximate the internal temperature of the potato at time $t = 3$.

Part b:

Using $\frac{d^2H}{dt^2}$, students were asked to determine whether the answer in part (a) is an underestimate or an overestimate of the actual internal temperature of the potato at time $t = 3$.

Part c:

An alternative model for the internal temperature of the potato was given by a function G satisfying the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$ with $G(t)$ measured in degrees Celsius and $G(0) = 91$. Students were asked to find an expression for $G(t)$ and, using this model, find the internal temperature at $t = 3$.

Comments on student responses and scoring guidelines:

Part a:

Three points were possible for students to earn. The first point was for the slope of the tangent line. This could be expressed with or without correct units as in $-\frac{1}{4}(91 - 27)$ or even $\frac{dy}{dx} = -16$. An incorrect slope could be used in an equation for the second point if that equation indicated a line passing through (0, 91). Thus, $y - 91 = -16x$ earned the first two points. The answer $y = 16x + 91$ earned only the equation point because the slope is incorrect, but the point (0, 91) is on this line. Using an incorrect slope could make it impossible for students to earn the third point, as in $\frac{dH}{dt} = -0.25(91 - 27) = -15 \rightarrow y = -15x + 91 \rightarrow y = 46$. This student earned one point for slope and one point for an equation, the arithmetic error coming off the third point. A common error was substituting 0 rather than 91 in the tangent line equation. An equation point could be earned using the slope $\frac{27}{4}$ obtained in this way, but the approximation was then 111.25 and did not earn the third point, 111.25 being greater than the initial temperature of the potato.

Part b:

$\frac{d^2H}{dt^2} =$ the equivalent of $\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H-27)$ or $\left(-\frac{1}{4}\right)\frac{dH}{dt}$ had to be calculated as well as be indicated as positive, and the answer "underestimate" had to be given. Common errors were referring only to "concave up" or discussing the second derivative at one point rather than on an interval. Students did not need to appeal to an interval, but they could not present a local argument showing $\frac{d^2H}{dt^2} > 0$ at only one point. And only one point is what students earned with all this work.

Part c:

There were five points available. The first point was for separating the variables and the second for the antiderivatives. Many students launched straight into calculating the antiderivative of $-(G-27)^{2/3}$, thus ignoring the variable t and earning zero points in part (c). The third point was for both showing a correct constant of integration and using (substituting) the initial condition $(t,G)=(0,91)$. A student's value of C, either correct or incorrect, could be used to earn the fourth point by showing an equation involving both G and t. The fifth point was for both solving for $G(t)$ and showing the value of $G(3)$, and could only be earned if the value of C was correct.

Observations and recommendations for teachers:

1. Students have to show work on this test. Even a simple calculation for slope needs to be shown and/or connected to the information given that will lead to that slope. Part (a) in this problem was scored more charitably. A simple presentation of $y=-16x+91$ earned both the slope and equation points. More work shown is sometimes needed, and has been, on the AP Calculus Exam. As simple as it seems, showing $\frac{dH}{dt} = -\frac{1}{4}(91-27)$ is an important display of knowledge because the basic calculus concept here is that the slope of a line tangent to the graph of H is given by $\frac{dH}{dt}$. It is clear from scoring hundreds of exams that some students do not make this connection. Teachers should require such work from their students.

2. Students should be aware of the context of these problems. Work resulting in an internal potato temperature of greater than 91 should be immediately suspect, and students should check back through their work if such a result arises from their work. This type of error will not be awarded a point.

3. When deciding whether a linear approximation is an over- or under- estimate, it should be clear from the concavity. However, the test often specifically asks for students to use the second derivative. It is the sign of the second derivative that determines the nature of the concavity. The word "concavity" can be ignored if the sign of the second derivative is correctly interpreted. Use of the proper concavity in determining the nature of the estimate is not sufficient to earn an explanation point in the absence of work showing the sign of the second derivative.

4. Many readers felt that students had difficulty in part (c) because no t showed in the given information. The first step in separating variables given $\frac{dy}{dt} = \text{expression}$ is to "multiply" both sides of the equation by dt. Sometimes this dt will be the only term showing on the right side of the equation. That led some students into more difficulties. There is an integrand of 1. There is something elementary, but important, going on here. The integrals $\int dt,\ \int dx,\ \int d\theta,$ and $\int d(uv)$ should be quickly, automatically, calculated by students. The integral $\int d(\text{anything}) = \text{anything } + C$. Not recognizing this simple idea put many students in the situation of earning 0 of 5 points in part (c).

5. Scoring the solving of a separable differentiable equation requires only one correct use of $+C$ even though technically there is a C on both sides of the equation after calculating antiderivatives. A point is earned for this $+C$ and using (substituting) the initial condition in the resulting equation. This is a recent change in the scoring. In the past a correct $+C$ earned a point and another point was awarded for using the initial condition. Arithmetically calculating the value of C incorrectly comes off the final answer point.

6. Students should carefully read the question posed. Students were asked to find in part (c) an expression for $G(t)$, which means to solve for $G(t)$. In this case, one point was awarded for preliminary work showing an equation involving $G(t)$ and t. In words, it was asked of students to calculate $G(3)$. Students should not worry about what work will award them points. But they should carefully read the question in order to make certain that all that has been asked for has been provided.