2016 BC – 4 Dennis Wilson, Landmark Christian School

Problem Overview:

Students were given a first-order non-separable differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

Part a:

Students were asked to find $\frac{d^2y}{dx^2}$ in terms of x and y.

Part b:

Students were given a point $(-2,8)$ on the graph of a particular solution to the differential equation. The students were asked if this point was a relative minimum, relative maximum, or neither.

Part c:

Students were given a point $(-1,2)$ for a different particular solution to the differential equation and asked to evaluate the limit

$$\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$$

Part d:

Students were given a point $(0,2)$ on a third particular solution to the differential equation and asked to approximate the value of the particular solution at $x = 1$ using Euler's method with two steps of equal size.

General scoring guidelines for the problem:

No Points were given for bald answers. While mistakes in part (a) affected parts (b) and (c), eligible incorrect answers could lead to full credit in these subsequent parts.

Part a: (2 points)

The first point was awarded for use of the chain rule. The second point was awarded for the successful substitution of the original differential equation into $\frac{d^2y}{dx^2}$. Students who made differentiation errors but applied the chain rule earned the first point. Students who incorrectly simplified the expression after a successful substitution lost the second point.

Common Errors – Students failed to use chain rule. Students did not substitute back in for $\frac{dy}{dx}$. Students made arithmetic or algebraic errors when simplifying the expression. Some students misread the question and thought that it asked for two separate expression, one in terms of x and another in terms of y. Students who did not substitute were still eligible for all the points on subsequent parts. Students who answered with any non-constant polynomial expression in terms of x and y were eligible for all points on parts (b) and (c).

<u>Part b:</u> (2 points)

To earn the first of two points, students should have expressed that $\frac{dy}{dx}$ is zero at the point and found the value of $\frac{d^2y}{dx^2}$ using an eligible form from part (a). Students then earned the second point for correctly concluding the value was a relative maximum. If students did not express that $\frac{dy}{dx} = 0$, then they could earn only one point with a correct value of $\frac{d^2y}{dx^2}$ and conclusion. If students made a mistake in the calculation of $\frac{d^2y}{dx^2}$, then they could earn one of the two points by stating $\frac{dy}{dx} = 0$ and giving a correct conclusion from the calculated value of $\frac{d^2y}{dx^2}$.

Common Errors – Students tried to justify the relative maximum using First Derivative Test. Some students tried to draw slope fields for justification. Students frequently wrote too much and worked themselves out of the last point. Students failed to show the first derivative was zero when using second derivative test. Students mixed up concave up/down and relative max/min when making their conclusion.

<u>Part c:</u> (3 points)

Students earned two of the first three points for each correct iteration of L'Hospital's Rule. Student must give some indication that a limit was being evaluated and that L'Hospital's Rule was being used. For failure to do each one of these, a point of the first two was deducted. An indication of a limit required a what (x), a where (-1), and an approach (→). This indication could happen early, late, or intermittently. An indication that L'Hospital's Rule was used included $\frac{0}{0}$, the word indeterminate, the word L'Hospital, or variations of the initials LH above the equal signs. The final third point was earned for the correct evaluation of $\left.\frac{g''(x)}{6}\right|_{(x,y)=(-1,2)} = -\frac{1}{3}$. This could include incorrect values stemming from eligible incorrect forms of $\frac{d^2y}{dx^2}$ in part a.

Common Errors – Students failed to differentiate the constant in the numerator. Students did not give some indication of either evaluating a limit or the use of L'Hospital's rule. Students used the quotient rule instead of L'Hospital's Rule. Students tried to solve the differential equation to substitute into the limit. Students did not connect $y = g(x)$ with the differential equation given in the stem.

Part d: (2 points)

To earn the first point, students should have the following evidence:

- Use of initial value, $h(0) = 2$
- Two complete iterations of Euler's method,
- A step size of $\frac{1}{2}$ in both iterations,
- The use of the problem's differential equation.

The second point was earned for the answer. Students could not earn this point without earning the first. If an explicit arithmetic mistake was found, students could still earn the first. If a table was used, the answer needed to be clearly identified at the end.

Common Errors – The most common errors from students who clearly knew Euler's method were from arithmetic mistakes. Students failed to use $\Delta x = \frac{1}{2}$. Students did not clearly identify the answer. Students performed more iterations than necessary, usually completing the third line where $x = 1$. Students used y'' for the slope instead of y'

Observations and recommendations for teachers:

(1) Teachers should always review the set of Free Response Questions from the previous year with their students. The evaluation of $\frac{d^2y}{dx^2}$ in terms of x and y was on the common differential equation question in 2015.

(2) The preeminence of the Second Derivative Test over the First Derivative Test cannot be stated enough.

(3) In teaching the Second Derivative Test, the importance that $\frac{dy}{dx} = 0$ should be emphasized. Students also should connect the sign of the second derivative with the concavity of the graph on the interval containing the critical point when applying the Second Derivative Test.

(4) Students need to be concise in answering questions. Brevity is important. A Table Leader joked that the exam should state "Justify your answer in twelve words or less."

(5) Students should know that L'Hospital's Rule can only be used for indeterminate ratios and that these indeterminate ratios should be clearly shown when it is used. Stating that L'Hospital's Rule applies is not a bad idea either.

(6) Students should be encouraged to explicitly state the answer in problems involving Euler's Method as $h(1) \approx 1.25$. It was frequently clear that students did not comprehend that Euler's Method produces an approximation and not the actual value of the function.

(7) While various methods to Euler's method are taught, students should make sure all the pieces of the method are clearly identifiable for the reader.