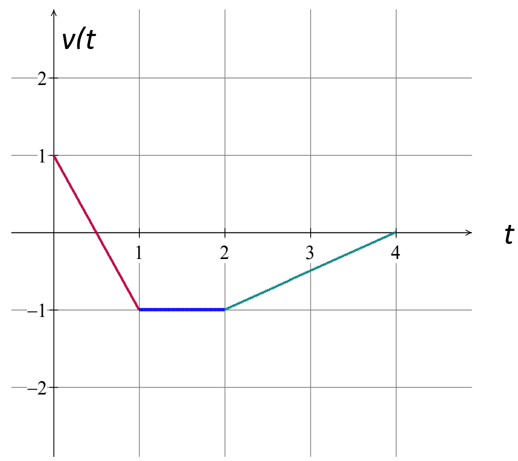
2016 BC-2 Dennis Wilson, Landmark Christian School

Problem Overview:

The problem involved the motion of a particle in which students were given an explicit expression for the horizontal velocity, $\frac{dx}{dt} = t^2 + \sin(3t^2)$, as well as a graph similar to the one at right, giving the vertical position of the particle. The position of the particle at $t = 0$ was given to be $(5,1)$.



Part a:

Students were asked to find the position of the particle at time $t = 3$.

Part b:

Students were asked to find the slope of the line tangent to the path of the particle at $t = 3$.

Part c:

Students were asked to find the speed of the particle at $t = 3$.

Part d:

Students were asked to find the total distance traveled by the particle from $t = 0$ to $t = 3$.

General scoring guidelines for the problem:

Part a: (3 points)

The first point in this problem was earned for a correct integral yielding the change in the horizontal position. Various forms of the integral were accepted, such as,

$$\int_0^3 \frac{dx}{dt}\,dt \text{ or } \int_0^3 \frac{dx}{dt} \text{ or } \int_0^3 x'(t)\,dt \text{ or } \int_0^3 x'(t).$$

The differential dt may be left off but if a differential was present, it must be dt to earn the point.

The second point was earned for the proper use of the initial condition. Students could earn the second point without the first by adding 5 to an incorrect integral as in the example below.

$$5 + \int_0^b \frac{dx}{dt}\,dt, \qquad 0 < b \leq 4$$

If students add 5 to the back end of the integral without a differential ($\int_0^3 x'(t) + 5$), then they would only earn the first and second point with a correct answer.

$\int_0^3 3t^2 + \sin(3t^2) + 5 = 14.377$ 1-1-1

$\int_0^3 3t^2 + \sin(3t^2) + 5 = 24.377$ 0-0-0

Any initial condition other than a numerical value of 5 did not earn this point.

Students could add the initial value after evaluating the definite integral with their calculator, but they lost a point if they linked unequal quantities while doing this.

$\int_0^3 3t^2 + \sin(3t^2)\,dt = 9.377 + 5 = 14.377$ 1-1-0 Linkage Error

The third point that students could earn was for the answer. Accepted answers are shown below:

$\left(14.377, -\frac{1}{2}\right)$ 1-1-1 with supporting work for first two points

$\left(9.377, -\frac{1}{2}\right)$ 1-0-1 with supporting definite integral

If answers were not given as an ordered pair, then $x(3)$ and $y(3)$ needed to be clearly labeled.

An indefinite integral could earn one of the first two points if the initial value is added to it and the correct answer is given. If the initial value is not added to the indefinite integral, then neither of the first two points are earned but the answer point can still be earned with $\left(14.377, -\frac{1}{2}\right)$.

Common Errors – Students frequently tried to evaluate the definite integral without technology. Some algebraic errors were made when equating definite integral with the change in x. Students frequently treated graph of y as graph of $\frac{dy}{dt}$ and used the enclosed area (definite integral) to find a change in y. Students tried to find a closed form for the function $x(t)$ and then solve for the constant of integration using the indefinite integral instead of the definite. Students who set the definite integral equal to a difference frequently made arithmetic errors when solving for $x(3)$. Students who used the net change made arithmetic errors at a much less frequent (almost never) rate.

<u>Part b:</u> (1 point)

One point was earned for the correct answer. The point was only given with supporting work.

$\frac{y'(3)}{x'(3)}$ or $\frac{\left(\left.\frac{dy}{dt}\right|_{t=3}\right)}{\left(\left.\frac{dx}{dt}\right|_{t=3}\right)}$ or $\left.\frac{dy}{dt}\cdot\frac{dt}{dx}\right|_{t=3}$

The answer could be .05 or .0502.

Minimal work earned the point with expressions such as

$\frac{\frac{1}{2}}{9+\sin 27}$ or $\frac{.5}{9.95638}$.

If the equation for line tangent to the path was given, the equation was read for the slope.

Common errors - Many students found the reciprocal of the slope, $\frac{dx}{dy}$. Students continued to misread the graph, using $y(3)$ instead of $y'(3)$. Some students differentiated $\frac{dx}{dt}$ and therefore used $\frac{d^2x}{dt^2}$ at $t = 3$ instead.

<u>Part c:</u> (2 points)

The first point was given for an expression of the speed at $t = 3$.

$\sqrt{\left(x'(3)\right)^2 + \left(y'(3)\right)^2}$ or $\left.\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\right|_{t=3}$

If no indication of the time $t = 3$ was given in the expression, then the first point could only be earned in the presence of the correct answer.

The second point was earned for the correct answer. The point for the answer could only be given if some evidence for the expression of speed was given. Incorrect values for $x'(3)$ and $y'(3)$ could be imported from part b and used without loss of points in part c if they were explicitly stated.

Common errors – Continuing to misread the graph, students sometimes used the value of $y(3)$ instead of $y'(3)$. If they had done so in part b, then they earned full points on part c. If not, then they lost points. Students lost both points if the expression for speed was not given. Student generally earned both points on this part. In general, only students with dashes or zeroes missed this part.

Part d: (3 points)

The first point was given for an integral expression of the distance from time $t = 0$ to $t = 2$. Students who did not give a single integral expression for the distance could still earn this point from a sum of two distance integrals. Some leeway was given to these students if they did not split the integral correctly so they could still earn one of the first two points.

The second point was earned for splitting the integrals at the time $t = 1$.

Some students split the integral but did not express the different values for $\frac{dy}{dt}$ in the integral. They earned full points for this if they clearly stated the values of dy/dt somewhere else as a piece-wise function. These students also earned this point if the numerical approximation for each integral was given in their response.

The third point given was for the answer.

Common errors – In general, students either knew how to split the integral or they did not. Students who did know how to split the integral sometimes lost a point for decimal presentation. While a missing third decimal place of zero in part b was accepted, it was not accepted in part d. A significant number of students wanted to treat the distance in the horizontal direction and the vertical direction separately.

Observations and recommendations for teachers:

(1) Students should be cautioned that information for the horizontal and vertical directions can be given in different forms. This could mean graphically and algebraically. It could also mean as position and velocity. In this problem, it was actually both.

(2) Students should identify the quantities related in a given graph and determine what information that the slope of the graph and area under the curve would represent.

(3) Students should be warned to not analytically evaluate definite integrals on calculator active questions, even when it is possible to find an antiderivative. The use of the calculator should be introduced early in the calculus course using applications so students will be familiar with the tool and when to use it.

(4) Students should learn to be concise in communicating expressions. Various notations for speed were given by students and sometimes they lost points by using ambiguous notation. For example, $\sqrt{x'(3)^2 + y'(3)^2}$ did not receive credit because it was questionable what quantity was squared.

Student who used the vertical bar to denote evaluation, $\left.\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\right|_{x=3}$, rarely if ever missed points froms notation.

(5) The Net Change theorem should be taught in some form. This is helpful when no closed-form expression of the position can be found using an antiderivative. It is also helpful in preventing simple arithmetic errors arising from carelessness.

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$