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Problem Overview:

Water is pumped into a tank at a rate given by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where t is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour. It is given that R is differentiable and decreasing on $0 \le t \le 8$. Some values of R are shown in the table below. It is also given that at time $t = 0$ there are 50,000 liters of water in the tank.

t (hours)	0	1	3	6	8
$R(t)$ (liters/hour)	1340	1190	950	740	700

Part a:

Students were asked to estimate $R'(2)$, showing the work that leads to the answer and indicating units of measure.

Part b:

Students were asked to estimate the total amount of water removed from the tank during the 8 hours using a left Riemann sum with four subintervals indicated by the table. Students were also asked if this was an over or under estimate of the water removed and to give a reason for their answer.

Part c:

Students were asked to find an estimate of the total amount of the water in the tank at the end of 8 hours, to the nearest liter, using the answer from part (b),

Part d:

It was asked whether there was a time for $0 \le t \le 8$ when the rate at which water is pumped into the tank is the same as the rate at which it is removed, and to explain why or why not.

Comments on student responses and scoring guidelines:

Part a:

In calculating this estimate, the interval $[1, 3]$ had to be used, and both a difference and quotient had to be seen in order to earn the first point. Thus, $\frac{R(3)-R(1)}{3-1}=-120$ or $\frac{950-1190}{2}$ earned the first point. The expression $\frac{R(3)-R(1)}{3-1}$ without the final answer was not acceptable. However, the final answer connected to this expression was considered evidence that values were correctly pulled from the table. The units point was earned only in the presence of an answer and had to be a correct version of L/hr/hr.

Part b:

There are four intervals in the table, and the widths of these intervals had to be shown in demonstrating an attempt at a left Riemann sum. The first point was for this demonstration and had to show a sum of products using 8 numbers, at least 7 of which had to be correct. If only 7 were correct, students did not earn the second point for showing the actual Riemann sum value, but were eligible for the third point in this part of the problem. Thus, $1340+2(1190)+3(950)+2(740)$ earned the first two points, the students not needing to simplify this answer and provide the number 8050. The expression $1340+2380+2850+1480$ does not show the multiplication by the correct interval widths, not enough work shown, and this earns only the second point, but not the first. Even in the presence of a right Riemann sum of 6710, students were eligible for the third point. The sum is always an overestimate when using a left sum for a decreasing function.

Part c:

The integral giving an estimate of the water removed from the tank was estimated in part (b). To get into this part of the problem for the first point, students had to show evidence of trying to calculate $\int_0^8 W(t)\,dt$.

The integral value is approximately 7836 liters. For the second point, students had to show appropriate use of this value, along with 50,000 and the estimate from part (b). Thus, use of an incorrect value imported from part (b) allowed students to earn the second point. Readers checked the arithmetic for this point.

Part d:

A nice way to deal with this part of the problem is to consider the function $f(t)=W(t)-R(t)$, noting that at the point where $t=0$ this is greater than 0 and noting also that $W(8)-R(8)<0$. Combining this information with the fact that $W(t)-R(t)$ is continuous or that IVT applies to this function verifies that at some point t on the interval $(0, 8)$ $W(t)-R(t)=0$. Not many students used this approach. Students who appeared to understand what was happening here often looked at $W(t)$ and $R(t)$ separately at the endpoints. These students had to identify the facts that $W(0)>R(0)$ and $W(8)<R(8)$ as well as the fact that BOTH these functions were continuous (or IVT applied) in order to earn both points. Consideration of the endpoint values did earn the first point, but most students using this approach to the problem did not earn both points in part (d). Students who appealed to the average value $\frac{1}{8}\int_0^8 W(t)\,dt \approx 979.52$ often noted, correctly, that this value is between $R(0)$ and $R(8)$. But since the time at which this average value is attained may not be the time at which the functions are equal, this argument earned zero points.

Observations and recommendations for teachers:

(1) This problem is a version of finding information about a total amount, given rates of input and output as well as an initial condition, the 50,000 liters. Examples of this type of problem are common on AP Calculus exams. Refer to 2010 AB/BC1, 2013 AB/BC1 or 2015 AB/BC1.

(2) The estimate of a derivative of a function, given tabular data values for the function, is common on AP Calculus exams. This estimate should be done by calculating the slope of a secant line using the narrowest interval in the table that includes the point in question, in this case in part (a) where $t = 2$. To show work, values from the table must be used in a rational expression for this secant line slope, showing some evidence of subtracting those values as well as the quotient. Although the numerical answer is easy to determine without pencil and paper, work must be shown on the AP Calculus exam.

(3) Knowing how to calculate a left, right, or mid-point Riemann sum is a requirement in the AB/BC curriculum. To show work, students need to show not just arithmetic results, but the use of interval widths as well as summing the products of those widths with appropriate function values. A reminder: Do Not Simplify arithmetic!! Some students had a nice setup, but made arithmetic mistakes in simplifying.

(4) Right and left Riemann sums are predictably either over- or under-estimates depending upon the function being increasing or decreasing. This should be illustrated and practiced in the classroom.

(5) Finding a total amount (as in part (c)) involves the initial amount, the amount removed, and the amount put in. These three quantities are found in different ways. But integrals of rates of change do calculate accumulated amounts. Since this was a calculator active problem, the amount of water that went into the tank is found using a definite integral functionality of a calculator. The setup, using correct mathematical notation, must be shown so that readers can see where this amount comes from. The point that students are awarded is for seeing that correct notation, rather than seeing the actual correct numerical amount of water. An error there comes off the final answer point.

(6) Asking about an intermediate value, as in part (d), requires students to invoke the IVT or the condition of continuity and to show two function values that as an interval contain the value in question. Part (d) was particularly difficult for students because looking at R and W separately required a more complicated argument. When two quantities are supposed to be equal at some point, it is not a bad idea to check the meaning of $R - W = 0$, as in this problem, and look at the implications of that equation. In particular, that equation gives us a new function to consider, $R - W$, with its own properties. Not many students looked at this function. A message to teachers might be to look at that form of any equation where it is desired to solve for x or t or whatever when $A = B$.

(7) An example of (6) above is provided by considering two functions that have the same derivative as in $f' = g'$. If we form the function $h = f - g$, we see that $h'(x) = f'(x) - g'(x) = 0$. By the MVT we can easily prove that if $h'(x) = 0$, then $h(x) = C$, a constant. (Note that this must be proven, not assumed merely because $\frac{d}{dx}(C) = 0$ because it is the **converse** of that fact). Since $h(x) = f(x) - g(x) = C$, we establish the fact that two functions with equivalent derivatives differ by a constant.