Exponential Modeling

By Jeff McCammon

Bacteria in a lab culture grows in such a way that the instantaneous rate of change of bacteria is directly proportional to the number of bacteria present. Suppose there are 3 million bacteria initially and five hours later, that number has doubled.

Find an exponential model that describes the growth of the bacteria.

Many textbooks encourage using the following model:

$y = Ce^{kt}$ where C represents the initial amount and k represents the constant of proportionality. In this case, we can solve directly for k by use of logarithms or special formulas.

For instance, in the example above, we could use the fact that at $t = 5$ hours, there will be 6 million bacteria. Consequently, by substitution,

$$6 = 3e^{5k}$$

$$2 = e^{5k}$$

$$\ln 2 = 5k$$

$$k = \frac{\ln 2}{5} \approx 0.1386$$

Yet, by observation, we can create a formula for $k = \frac{\ln M}{t_0}$, where M is the multiplier of growth and t_0 is the time it takes to multiply the growth population.

However, I propose there is a form for exponential growth models that may be more intuitive to student understanding. The following is the formula:

$$y = C(M)^{t/t_0}$$

where C is the initial amount, M is the multiplier of growth and t_0 is the time it takes to multiply the growth population.

For the above example the growth model would be:

$$y = 3(2)^{t/5}$$

This form preserves the initial problem situation and can easily be checked for accuracy. If a student has to reconfigure the formula into the standard form, the conversion of $M = e^{\ln M}$ quickly can be substituted for the multiplier of growth. Thus,

$$y = 3e^{\left(\frac{ln2}{5}\right)t}$$

This conversion also affirms the value of $k = \frac{\ln M}{t_0} = \frac{\ln 2}{5}$.

This secondary form can also be used when the multiplier is implied in the context of the problem. Notice the following example:

The population of a city is initially 4,000. In 3 years, the city grows to 4,200. Find an exponential growth model.

We will proceed as usual except define the multiplier, $M = \frac{Final\ Population}{Initial\ Population} = \frac{4{,}200}{4{,}000}$

Thus the model of growth is

$$y = 4{,}000\left(\frac{4{,}200}{4{,}000}\right)^{t/3}$$

Or in standard form:

$$y = 4{,}000e^{(\ln(4{,}200/4{,}000)/3)t}$$

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