Problem Overview:

Students are given the function $f(x) = \dfrac{1}{x^2 - kx}$ where k is a non-zero constant. It is also given that the derivative of f is $f'(x) = \dfrac{k - 2x}{\left(x^2 - kx\right)^2}$.

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Part a:

Given that $k = 3$ so that $f(x) = \dfrac{1}{x^2 - 3x}$, students were asked to write an equation of the line tangent to the graph of f at the point where the x-coordinate is 4.

Part b:

Given that $k = 4$ so that $f(x) = \dfrac{1}{x^2 - 4x}$, students were asked to determine whether f has a relative max, a relative min, or neither at the point where $x = 2$ and justify the answer.

Part c:

Students were asked to find the value of k for which f has a critical point at $x = 5$.

Part d:

Given that $k = 6$ so that $f(x) = \dfrac{1}{x^2 - 6x}$, students were asked to find the partial fraction decomposition of f and find $\int f(x)\,dx$.

Comments on Scoring:
Part a:

The calculus involved in finding an equation of a tangent line is using $f'(4)$. The calculation is not difficult, and therefore the mere presentation of a correct equation earned both points in this part of the problem. An incorrect slope **shown to be coming from** f' made the student eligible for the second point by presenting an equation that readers had to verify as correctly using both $x = 4$ and $y = \dfrac{1}{4}$.

Part b:

Consideration of f' close to $x = 2$ earned the first point. Thus, the statement "a relative max because f' changes from + to – at $x = 2$" earned both the first and second points (the second point requiring both the answer and a justification). Students did not need to show explicitly that $f'(2) = 0$. A student using the second derivative test had to include the fact that $f'(2) = 0$ along with a correct calculation leading to $f''(2) = -\frac{1}{8}$. The statement "relative maximum because f changes from increasing to decreasing at $x = 2$" was not acceptable because there is no link to the derivative in that statement. We have in that statement the answer, but we do not have the justification.

Part c:

The only possible value of k comes from $x = -5$ and setting the numerator of $f' = 0$. This gives $k = -10$, which if it was the only thing shown on a paper earned the one point in this part of the problem. A difficulty in grading this part of the question arose when students also considered the possibility of the derivative not existing. In this case, correct calculation leads to $k = -5$. However, the function is not defined for the situation in which both $x = -5$ and $k = -5$. Because there was only one point available in this part of the problem, students who correctly found both values but failed to exclude the –5 were awarded the point.

Part d:

There were four points in this part of the problem, two for the partial fraction decomposition and two for the general antiderivative. A correct factorization showing $\frac{A}{x} + \frac{B}{x-6}$ earned the first of the two decomposition points. Correct values for both A and B earned the second of these points. An incorrect factorization earned neither of these two points, but this was eligible for the two general antiderivative points if of the form $\frac{\text{constant}}{\text{linear}} + \frac{\text{constant}}{\text{linear}}$ where the linear denominators were different. The first of the antiderivative points was earned for one correct logarithmic term including absolute values. The second antiderivative point was earned for the remainder of a correct answer including $+C$.

Observations and recommendations for teachers:

(1) Finding an equation of a line tangent to a function is a fundamental skill that has nothing to do with the complexity or rational nature of f in this problem. Students should practice this using $y - \_\_ = \_\_(x = \_\_)$ and fill in the blanks. The only calculus in play here is using f' for the slope. Students should be coached to show some work calculating the slope, as a connection to f' is important and should be shown as work leading to this answer.

(2) In checking for a relative max or min (looking for critical points) it is important to examine both $f'=0$ and where f' does not exist. A rational function is a case (this should probably be taught as a necessary part of the analysis in all cases) where it is important to look back at the original function and determine if results obtained by examining f' yield values of x that are in the domain of f.

(3) A search for relative extrema involves finding a value for x AND noting a sign change on an interval containing x. It is not sufficient to check one value to the left and one value to the right. Any statement justifying the existence of a max or min must use language that indicates an interval to the left and an interval to the right of the value.

(4) Partial fraction decomposition is not often on the AP Calculus exam free response section. Note that in this problem, the decomposition is of the most basic variety: a simple factorization into linear factors. While this should be taught and practiced in more complicated scenarios as well, certainly a linear factorization should be a basic skill for BC students. This can also be done using the "Heavyside" or "cover up" method. As of this writing, January 2016, a helpful article explaining this method can be found at
http://math.mit.edu/~jorloff/suppnotes/suppnotes01-01a/ Choose 01f.pdf.
There is also a useful article on Wikepedia.

(5) Students need to be reminded that an indefinite integral gives a general antiderivative and requires a $+C$. The antiderivative of $\dfrac{1}{\text{linear}}$ is a natural log (use absolute value). This should be practiced in simple cases such as $\displaystyle\int \frac{dx}{x-7} = \ln|x-7| + C$ (note the absolute value) and in cases such as $\displaystyle\int \frac{dx}{3-5x} = -\frac{1}{5}\ln|3-5x| + C$ where, in the latter example, a bit of a u-substitution is required.