MATH 2242 **Taylor Series ("Maclaurin" Series if centered at 0)**

A Taylor series is a power series which can be derived for a function which has derivatives of all orders. The key is knowing how to calculate the coefficient of each term.

Example 1: The Taylor series centered at 0 (therefore sometimes called a "Maclaurin" series) for the famous function e^x is $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$. We can clearly see the powers of x. How are the coefficients derived?? If we start counting at $k=0$ we have $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...=\sum_{k=0}^{\infty}\frac{x^k}{k!}$.

The coefficients are found using derivatives of e^x evaluated at 0 and, as you can see, factorials dividing.

If $f(x)=e^x$, then $f'(x)=e^x$, $f''(x)=e^x$, $f^{(3)}(x)=e^x$, $f^{(4)}(x)=e^x$, etc.….(what a friendly function!!!!)
Since the center is 0, we evaluate each of these at $x=0$ and get $e^0=1$. Any coefficient can be calculated by taking $e^0=1$ and dividing by the appropriate factorial (which in number always matches the power of x).

Thus a formula for the coefficient of x^n is $\frac{f^{(n)}(0)}{n!}$. That's all there is to it !!!

Example 2: Derive a Taylor series for $\sin(x)$ centered at 0. (This is the "Maclaurin" series for $\sin(x)$).

First we calculate some derivatives of $f(x)=\sin(x)$ and then we substitute 0 for x:

$f^{(0)}(x)=\sin(x) \rightarrow f^{(0)}(0)=\sin(0)=0$
$f^{(1)}(x)=\cos(x) \rightarrow f^{(1)}(0)=\cos(0)=1$
$f^{(2)}(x)=-\sin(x) \rightarrow f^{(2)}(0)=-\sin(0)=0$
$f^{(3)}(x)=-\cos(x) \rightarrow f^{(3)}(0)=-\cos(0)=-1$
$f^{(4)}(x)=\sin(x) \rightarrow f^{(4)}(0)=\sin(0)=0$

Can you see a pattern? *Can we stop*????

Now to construct the series: for example, the coefficient of x^3 is $\frac{f^{(3)}(0)}{3!}=\frac{-1}{3!}$.

Notice that the coefficients for all the even powers of x are 0. Use these with matching powers of x.

Therefore, $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

Example 3: We have already seen Taylor series for sin(x) centered at 0. Since cos(x) is the derivative of sin(x), we can take the derivatives of the terms in Example 2:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

A series at a non-zero center is derived similarly using derivatives and factorials.

Example 4: Derive a series for sin(x) centered at $x = \frac{\pi}{6}$.

We need derivatives evaluated at $\frac{\pi}{6}$, factorials, and this time powers of $\left(x - \frac{\pi}{6}\right)$.

We ALWAYS use $\left(x - center\right)$.

$$f^{(0)}(x) = \sin(x) \;\rightarrow\; f^{(0)}\left(\tfrac{\pi}{6}\right) = \sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2}$$
$$f^{(1)}(x) = \cos(x) \;\rightarrow\; f^{(1)}\left(\tfrac{\pi}{6}\right) = \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}$$
$$f^{(2)}(x) = -\sin(x) \;\rightarrow\; f^{(2)}\left(\tfrac{\pi}{6}\right) = -\sin\left(\tfrac{\pi}{6}\right) = -\tfrac{1}{2}$$
$$f^{(3)}(x) = -\cos(x) \;\rightarrow\; f^{(3)}\left(\tfrac{\pi}{6}\right) = -\cos\left(\tfrac{\pi}{6}\right) = -\tfrac{\sqrt{3}}{2}$$
$$f^{(4)}(x) = \sin(x) \;\rightarrow\; f^{(4)}\left(\tfrac{\pi}{6}\right) = \sin\left(\tfrac{\pi}{6}\right) = \tfrac{1}{2}$$

Can you see a pattern? *Can we stop* ????

Now to construct the series: for example, the coefficient of $\left(x - \frac{\pi}{6}\right)^3$ is $\frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!} = \frac{-\frac{\sqrt{3}}{2}}{3!} = -\frac{\sqrt{3}}{2 \bullet 3!}$.

Thus the series is $\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2 \bullet 2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2 \bullet 3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{2 \bullet 4!}\left(x - \frac{\pi}{6}\right)^4 \cdots\cdots$

If we calculate the derivative of this series, what do we get and what function does this series represent?

1. Calculate the first 5 terms of a Taylor series for $\ln(x)$ centered at 1. Use this result to write the first 5 terms of a series for $\frac{\ln(x)}{x}$.

2. Calculate the first 4 non-zero terms of a Taylor series for $\cos(x)$ centered at $x = \frac{\pi}{4}$.

3. A Maclaurin series for a function $f(x)$ is given by $f(x) = 2x^2 - \frac{x^4}{2!} + \left(\frac{1}{4!} - \frac{1}{3!}\right)x^6 - \frac{x^8}{6!} \cdots\cdots$

Calculate the value of the 6$^{\text{th}}$ derivative of f at $x = 0$ (that is, calculate $f^{(6)}(0)$).