MATH 2242 **Power Series and more help from "geometric"**

I. A geometric series is formed by adding terms starting with a_1 then $a_1 r$, $a_1 r^2$, $a_1 r^3$, etc.

We know that if $-1 < r < 1$ there is a finite "infinite sum" given by $S_\infty = \dfrac{first\ term}{1-r} = \dfrac{a_1}{1-r}$.

Example 1: The fraction $\dfrac{1}{1-x}$ can be thought of as a geometric series with $a_1 = 1$ and $r = x$.

We can write $\dfrac{1}{1-x} = \dfrac{first\ term}{1-r} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$.

This is an example of a **power series** which is a sum of whole number powers of x with coefficients:

$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$ centered at $a = 0$. Where is a???....... not showing because it's 0.

The series $c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x-2)^n$ is also a power series. This one is centered at $a = 2$.

Example 2: What about a power series for $\dfrac{1}{1-3x}$? We can use $a_1 = 1$ and $r = 3x$.

Or we can use $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ from example 1 and substitute $3x$ for x.

The result is $\dfrac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + (3x)^4 + \cdots = 1 + 3x + 9x^2 + 27x^3 + 81x^4 + \cdots = \sum_{n=0}^{\infty} 3^n x^n$

For what values of x does this series converge (have a finite infinite sum)? We must have $|r| < 1$ because this is a geometric power series. Therefore $-1 < 3x < 1 \rightarrow -\dfrac{1}{3} < x < \dfrac{1}{3}$.

If x is any other value than those in the interval $-\dfrac{1}{3} < x < \dfrac{1}{3}$, we cannot guarantee convergence.

Therefore, it is not really true that $\dfrac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + (3x)^4 + \cdots$ unless $-\dfrac{1}{3} < x < \dfrac{1}{3}$.

II. Not only can we substitute into a known series to create a new one, we can also do some calculus which means we can either integrate or differentiate the terms. This leads to some interesting, more complicated, and NOT geometric series.

Example 3: $\dfrac{1}{1+x}=\dfrac{1}{1-(-x)}=\dfrac{\textit{first term}}{1-r}$ which means that $a_1=1$ and $r=-x$. We start with 1 and multiply by $-x$ to produce the series $\dfrac{1}{1+x}=1-x+x^2-x^3+x^4+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n$.

(NOTE: This only converges if $|r|<1$ which means that $-1<x<1$ and therefore $1+x>0$.)

Now we use this information and $\int\dfrac{1}{1+x}dx=\ln(1+x)+C$ to produce a power series for $\ln(1+x)$

$\int\dfrac{1}{1+x}dx=\int\left(1-x+x^2-x^3+x^4+\cdots\right)dx=\int\sum_{n=0}^{\infty}(-1)^n x^n\,dx$. This gives us a new power series for $\ln(1+x)$:

$\ln(1+x)=\int\dfrac{1}{1+x}dx=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+1}}{n+1}$ (And, oh yeah……there's a $+C$ somewhere).

Now we know that $\ln(x+1)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$

NOTES:
- (i) This is NOT geometric……. there is no r value
- (ii) We do not need absolute value for the $\ln$ because $1+x>0$

Other important series which are NOT geometric such as $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\cdots$ must be derived using a different method than that in example 3 above. We will call these "Taylor Series" and show how to derive them for functions such as $\sin(x)$ and $\cos(x)$ using derivatives.

Class discussion: Using geometric series properties and some calculus,……….

1. (a) Produce a series for $\frac{1}{1+x^2}$ by finding a_1 and r.

(b) Use your result in part (a) and some calculus to find a power series for $\tan^{-1}(x)$.

(c) What is a general term for the series in part (a)? Use this to get a general term for the series calculated in part (b).

2. (a) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ as stated above, find a power series for e^{-x}.

(b) What is a power series for $e^x + e^{-x}$? ...... $e^x - e^{-x}$?

(c) What is a general term, starting at $n = 0$, for your series in parts (a) and (b) above?

3. Using the series for e^x in #2a above, find a series for
[NOTE: we will use some substitution, some calculus, or both].

e^{2x}

$e^{(x-1)}$

$e^{(x-1)^2}$

$\int e^{(x-1)^2}\, dx$

3. The power series for cos(x) is $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$.
(This is the Taylor Series centered at $a = 0$ also known as the MacLaurin series because of 0 as the center).

(a) Using some calculus, derive a series for sin(x).

(b) Using substitution, derive a series for $\sin(3x)$ and $\cos(3x)$.

(c) How could we get a series for $\sin(3x)\cos(3x)$??