Calculus II
Polar Coordinates – I

- Summary of important concepts

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[1] **<u>Polar Coordinates</u>** are used to plot points on a rectangular coordinate system. The two coordinates give the distance from the Origin (the pole) and the angle θ with respect to the positive *x*-axis. Because *r* is used to describe the distance from the pole, we have the following:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad \theta = \tan^{-1}\left(\tfrac{y}{x}\right) \text{ (Check Quadrant for } \theta\text{)}$$

Example 1: The polar coordinates $\left(2, \frac{\pi}{6}\right)$ are coordinates for the point $\left(2\cos\left(\frac{\pi}{6}\right), 2\sin\left(\frac{\pi}{6}\right)\right) = \left(\sqrt{3}, 1\right)$.

Example 2: The polar coordinates $\left(-2, \frac{\pi}{6}\right)$ locate the point "backwards" from the direction $\theta = \frac{\pi}{6}$.
When *r* is negative, we "aim" in the direction of the given θ but move "backwards" from the Origin.
Add π to θ: this point is found in Quadrant III and is the same as the polar point $\left(2, \pi + \frac{\pi}{6}\right) = \left(2, \frac{7\pi}{6}\right)$.
The rectangular coordinates are $\left(-2\cos\left(\frac{\pi}{6}\right), -2\sin\left(\frac{\pi}{6}\right)\right) = \left(-\sqrt{3}, -1\right)$ which is in Quadrant III.

Example 3: Given rectangular coordinates $\left(-2, 2\sqrt{3}\right)$, we have a point in Quadrant II.
Thus the angle θ is such that $\frac{\pi}{2} < \theta < \pi$. The angle $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.
Since this is not in Quadrant II, we use $\theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$ which IS in Quadrant II.
We can easily find $r = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} = 4$.

[2] **<u>Equations in Polar Coordinates</u>**: Polar equations are often useful in expressing curves which are not functions of *x*. As a result, simple functions of *x* sometimes look complicated in polar coordinates, while very complicated expressions involving *x* and *y* can be elegantly expressed using polar coordinates.

Example 1: The line $2x + y = 3$ becomes

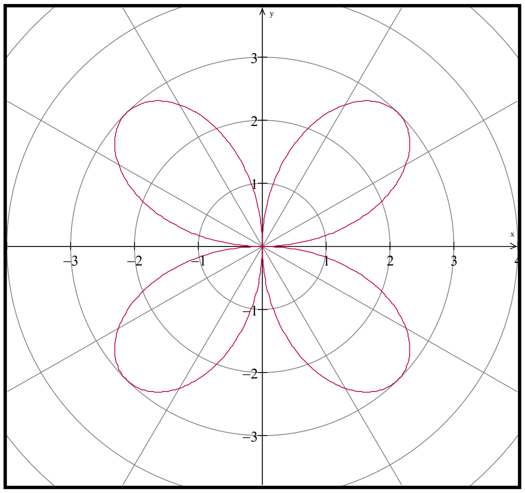
$$2r\cos\theta + r\sin\theta = 3 \;\rightarrow\; r = \frac{3}{2\cos\theta + \sin\theta}.$$

Example 2: The four loops defined by $r = 3\sin(2\theta)$ cannot be expressed as one function of *x*. The graph is at right.
Notice that the graph does not pass the vertical line test.
If we multiply both sides by r^2 we get the following:

$$r^3 = 3r^2\sin(2\theta) = 3r^2\left(2\cos\theta\sin\theta\right) = 6r\cos\theta r\sin\theta$$

This is equivalent to $\left(\pm\sqrt{x^2 + y^2}\right)^3 = 6xy$.

(NOTE that $\pm$ is needed in order to account for points in Quadrants II and IV where only one of *x* or *y* is negative. This accounts for different points when *r* is negative.)
Obviously this would be difficult to plot using the rectangular coordinate expression.



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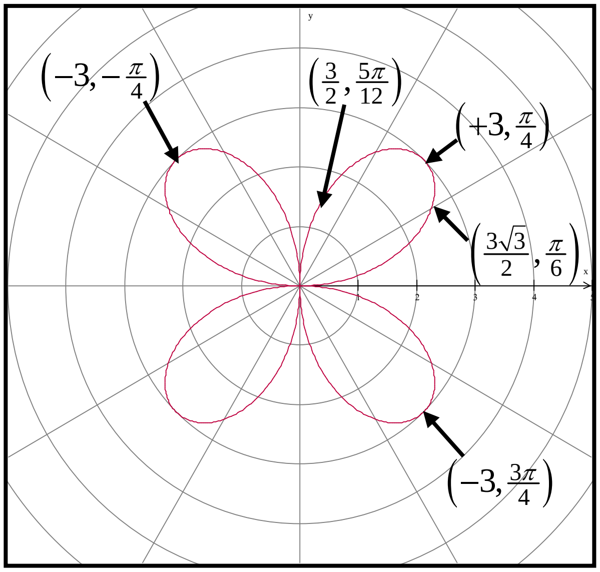
[1] **Graphing using polar coordinates:**



Another look at $r = 3\sin(2\theta)$ and this time we plot some points (shown on the graph at right). Notice that if $\theta = \frac{\pi}{4}$, we have $r = +3$, but if $\theta = -\frac{\pi}{4}$, we have $r = -3$ which locates the point in Quadrant II rather than IV. (We "aim" at $\theta = -\frac{\pi}{4}$ but go "backwards" from the Origin into Quadrant II because *r* is negative).

The entire graph can be generated on a graphing calculator using either $-\pi < \theta < \pi$ or $0 < \theta < 2\pi$.

A table of values of *r* for some special values of θ is shown at right. The points for which $0 < \theta < \frac{\pi}{2}$ trace the loop in Quadrant I. Notice that for $\frac{\pi}{2} < \theta < \pi$ the values of *r* are negative. These points trace the loop in Quadrant IV because *r* is negative, locating the point "backwards" from the original values of θ which are in Quadrant II.

In order to plot the points defining the loops in Quadrants II and III, we use values of $\pi < \theta < 2\pi$. These values are not shown in the table.

Notice in the table that values for both θ and 2θ are given. Be certain to plot points based only on the values of θ. The values of 2θ are shown to facilitate calculations for $r = 3\sin(2\theta)$.

In order to plot the **entire curve** in polar coordinates, be certain to examine all values for $0 < \theta < 2\pi$.

$r = 3\sin(2\theta)$	θ	2θ
0	0	0
$\frac{3}{2}$	$\frac{\pi}{12}$	$\frac{\pi}{6}$
$\frac{3\sqrt{3}}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
3	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\frac{3\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
$\frac{3}{2}$	$\frac{5\pi}{12}$	$\frac{5\pi}{6}$
0	$\frac{\pi}{2}$	π
$-\frac{3}{2}$	$\frac{7\pi}{12}$	$\frac{7\pi}{6}$
$-\frac{3\sqrt{3}}{2}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$
-3	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$
$-\frac{3\sqrt{3}}{2}$	$\frac{5\pi}{6}$	$\frac{5\pi}{3}$
$-\frac{3}{2}$	$\frac{11\pi}{12}$	$\frac{11\pi}{6}$
0	π	2π

[2] **More complicated graphing: Example 1**

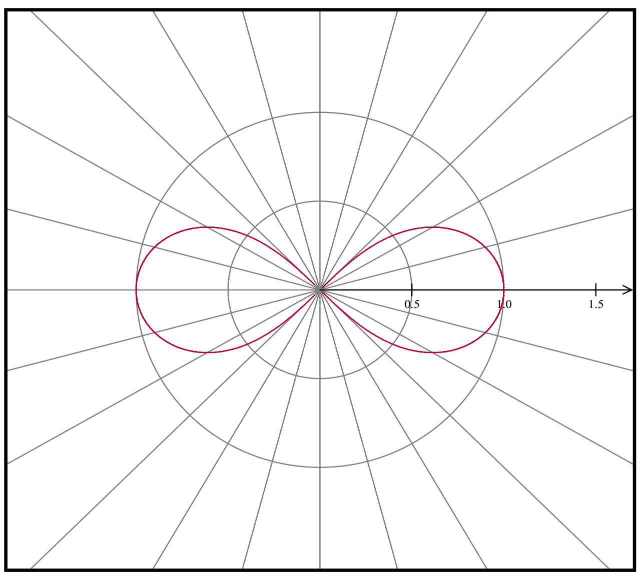
Suppose that $r^2 = \cos(2\theta)$. Since $r = \pm\sqrt{\cos(2\theta)}$, the values of r can only be found from positive values of $\cos(2\theta)$, where the values of θ are $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$, etc. There are no values of r for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$, etc.

To plot the curve $r^2 = \cos(2\theta)$ in polar coordinates, it is sufficient to plot $r = +\sqrt{\cos(2\theta)}$ for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$ generating the right and left portions of the curve respectively.

(NOTE: when using a graphing calculator, you may have to experiment with small values of "θstep" in order to plot points near the Origin; also, you should experiment with different values for "θmax" and "θmin" to see which generate the right or left parts of the curve).

The complete graph of this **lemniscate** $r^2 = \cos(2\theta)$ is shown in the polar graph below.
(In *Calculus* by Thomas 12th ed., see Example 2 page 633 for a similar graph discussed in more detail).

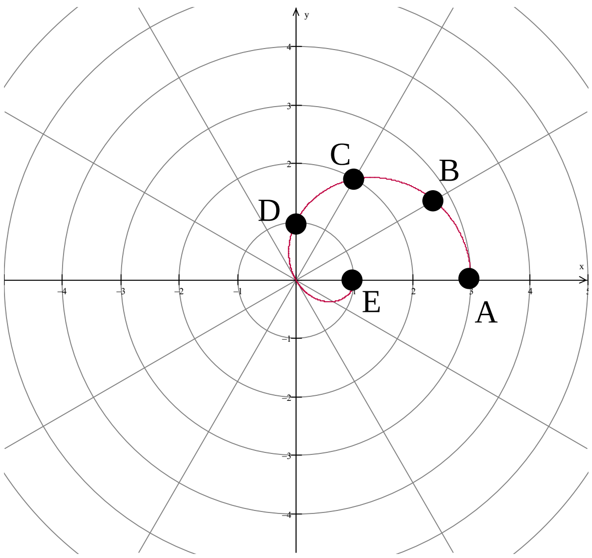
$r^2 = \cos(2\theta)$



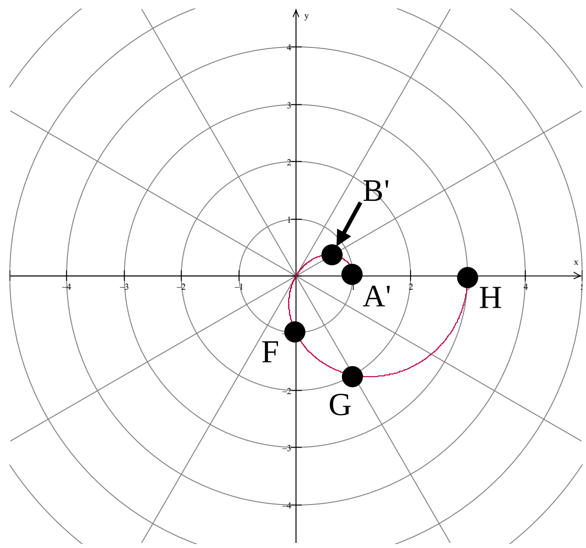
More complicated graphing: Example 2

Two graphs are shown below only for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$. The points A and A' correspond to $\theta = 0$ and $\theta = \pi$ respectively. The point E is the same as the point A'.

$r = 2\cos(\theta) + 1$ *for* $0 \le \theta \le \pi$



$r = 2\cos(\theta) + 1$ *for* $\pi \le \theta \le 2\pi$



Points below are given as (r, θ), and remember that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ for rectangular.

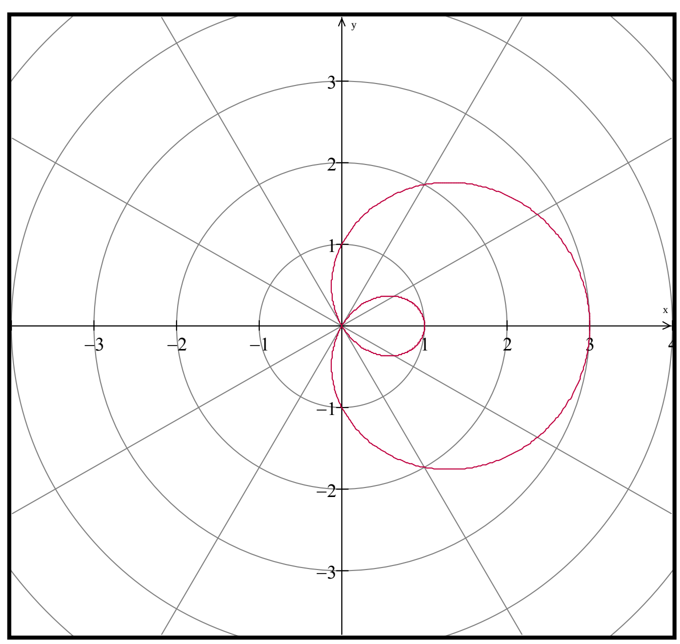
A: Polar (3, 0)
Rectangular (3, 0)

A': Polar $(-1, \pi)$
Rectangular (1, 0)

B: Polar $\left(\sqrt{3}+1, \frac{\pi}{6}\right)$
Rectangular $\left(\frac{3+\sqrt{3}}{2}, \frac{\sqrt{3}+1}{2}\right)$

B': Polar $\left(1-\sqrt{3}, \frac{7\pi}{6}\right)$
Rectangular $\left(\frac{3-\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right)$

The complete graph for $r = 2\cos(\theta) + 1$ is shown at right. Notice that it is the combination of the graphs above. Some points are the same, but located for different values of θ. The complete graph is found using $0 \le \theta \le 2\pi$.
This curve is called a **limaçon**.



Exercise: Find both the polar and rectangular coordinates of the points C, D, E, F, G, and H in the graphs on this page above.
Answers are provided below.

Answers to exercise on above page:

	C	D	E	F	G	H
Polar	$\left(2, \frac{\pi}{3}\right)$	$\left(1, \frac{\pi}{2}\right)$	$\left(-1, \pi\right)$	$\left(1, \frac{3\pi}{2}\right)$	$\left(2, \frac{5\pi}{3}\right)$	$\left(3, 2\pi\right)$
Rect	$\left(1, \sqrt{3}\right)$	$\left(0, 1\right)$	$\left(1, 0\right)$	$\left(0, -1\right)$	$\left(1, -\sqrt{3}\right)$	$\left(3, 0\right)$