2015 BC6 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given a Maclaurin series for a function f: $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^2 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and told that this series converges to $f(x)$ for $|x| < R$ where R is the radius of convergence of the series.

Part a:

Students had to use a ratio test to find R.

Part b:

Students had to write the first four non-zero terms of the Maclaurin series for f' and express f' as a rational function for $|x| < R$.

Part c:

Students were asked to write the first four non-zero terms of the Maclaurin series for e^x and use this series to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Part a:

The basis of the ratio test involves comparing limits of $\left|\frac{a_n}{a_{n+1}}\right|$ or $\left|\frac{a_{n+1}}{a_n}\right|$ to 1. Many students used absolute value later, if at all. The absolute value issue was dealt with more specifically for the third point in part (a). For the first point, readers looked for a correct ratio of coefficients in either form $\frac{a_n}{a_{n+1}}$ or $\frac{a_{n+1}}{a_n}$.
Without absolute value (often), these setups had to correctly deal with parentheses. For example, $\frac{(-3)^n}{n+1} \bullet \frac{n}{(-3)^{n-1}} = \frac{-3^n}{n+1} \bullet \frac{n}{3^{n-1}}$ are correct while $\frac{(-3)^n}{n+1} \bullet \frac{n}{3^{n-1}}$ and $\frac{-3^n}{n+1} \bullet \frac{n}{(-3)^{n-1}}$ are incorrect.
The second point was for indicating and calculating a limit from an appropriate ratio. To indicate a limit, both n and ∞ had to be used. Students' calculated limits had to be consistent with the initial setup.
The determination of the radius of convergence for the third point could not contain arithmetic or algebraic errors. Misuse of absolute value penalized this third point. For example, $-\frac{1}{3} < x < \frac{1}{3}$ is correct and in the presence of $R = \frac{1}{3}$ earns the third point; but even with the correct answer, work showing $-\frac{1}{3} > x > \frac{1}{3}$ would make the student ineligible for the third point. A correct inequality without specifying $R = \frac{1}{3}$ did not earn the third point.

Part b:

The first two points were for the first four non-zero terms of f'. Students could earn one of these points with 3 of the 4 terms correct. Readers only looked at the first four terms and not at any subsequent terms. The terms could be presented in a list, and coefficients did not need to be simplified. The third point was earned by showing $\frac{1}{1+3x}$ or an equivalent form.

Part c:

The first point was earned for the first four correct terms for the Maclaurin series for e^x. Readers did not look beyond four correct terms. In order to earn the second and/or third points, students must have earned the first point. To earn the second and/or third points students must have combined like terms by expressing a single coefficient for each distinct degree of x. Thus, $T_3 = x + \left(1 - \frac{3}{2}\right)x^2 + \left(\frac{1}{2} - \frac{3}{2} + 3\right)x^3$ was acceptable but $T_3 = x - \frac{3}{2}x^2 + 3x^3 + x^2 - \frac{3}{2}x^3 + \frac{x^3}{2!}$ was not acceptable. Expressions of degree higher than three (students were specifically asked for the "third degree") could earn one point in the presence of the three requested terms and no constant term. Note that this method of scoring the test does not involve simplification of the coefficients, rather the combining of parts of each coefficient for each order of x.

Observations and recommendations for teachers:

(1) A ratio test can be started by correctly examining $\frac{a_n}{a_{n+1}}$ or $\frac{a_{n+1}}{a_n}$. This should be practiced using terms that include negative values. The absolute value that should be a part of a correct calculation of the limit needs to be resolved somehow in a correct inequality involving x in order to find the radius of convergence. Terms of a series are often presented using terms involving $(-1)^{n \text{ or } n+1 \text{ or } n-1}$ rather than, as in this problem, –3. Students should see examples expressed in both manners.

(2) According to the BC curriculum, a few series need to be known: e^x, $\sin(x)$, $\cos(x)$ and $\frac{1}{1-x}$.

(3) Teachers should relate the series for $\frac{1}{1-x}$ to geometric series as in $a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots = \frac{a_0}{1-r}$.
A vast majority of student work showed no knowledge of this series. Student answers to this in part (b) usually had no rational function or expressed another series of some kind. This may also imply that the meaning of "rational expression" needs to be emphasized in the classroom.

(3) Manipulating series, including substitution, term-by-term differentiation and integration, and multiplication of two series, needs to be practiced. Students showed difficulties in distributing while multiplying two series. Combining coefficients for each order of x is important in clearly expressing a Taylor polynomial. 2011 BC6 part (c) provides an example (although this does not involve multiplication of series, it does involve combining parts of a coefficient).