Problem Overview:

Students were given the curve defined by $y^3 - xy = 2$ and the fact that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

Part a:

Students were asked to write an equation for the line tangent to the curve at $(-1, 1)$.

Part b:

Students had to find all points on the curve at which the line tangent to the curve is vertical.

Part c:

Students were asked to evaluate $\frac{d^2y}{dx^2}$ at the point where $x = -1$ and $y = 1$.

Part a:

The first point of two possible points was for calculation of the slope using the given derivative. Of course, no simplification is required meaning that $\frac{1}{3(1)^2 - (-1)}$ earns the first point. The second point was for an equation of the tangent line. If students had work, but did not earn the first point, an equation using the incorrect slope calculated AND containing the point $(-1, 1)$ did earn the second point.

Part b:

The first point was for setting $3y^2 - x = 0$ which could be accomplished by writing $y^3 - \left(3y^2\right)y = 2$, because we must have $3y^2 - x = 0 \rightarrow x = 3y^2$. An algebraic expression was required in order to earn the first point. A verbal description indicating setting the denominator of the derivative equal to zero was not accepted for the first point. Since the second point required an equation in one variable, the equation $y^3 - \left(3y^2\right)y = 2$ earned both the first and second points. Note that this equation implies that $y = -1$ which implies that $x = 3$. Other correct expressions earning the second point such as $\left(-\sqrt{\frac{x}{3}}\right)^3 - \left(-\sqrt{\frac{x}{3}}\right)x = 2$ were shown to readers during the training for scoring this part of the problem. To be eligible for the third point, students must have shown that $(3, -1)$ is on the curve and have earned the first point.

Part c:

For the first two points, students had to be using a quotient rule. Missing a denominator earned zero of these two points. Presentation errors involving parentheses, but an otherwise correct quotient rule, earned one of the two points. Thus a differentiation such as $\frac{\left(3y^2-x\right)\frac{dy}{dx}-y6y\frac{dy}{dx}-1}{\left(3y^2-x\right)^2}$ earned one of the first two points.

Any differentiation error cost students one point. Students were eligible for the third point if either $\frac{dy}{dx}$ or y' was present either in a correct or incorrect derivative. The third point was earned for a correct substitution for every $\frac{dy}{dx}$ or y'. Students were eligible for the answer point, the fourth point, if there were no differentiation errors, and the third point had been earned.

Observations and recommendations for teachers:

(1) Students should practice using a given differential equation as a way to get the slope for a tangent line equation. Using $y-y_1=m\left(x-x_1\right)$ is a VERY good idea because students will have been taught other methods involving solving for "b" in the classic slope-intercept form $y=mx+b$, and subsequently do some unnecessary and sometimes unusual work in trying to solve for b. A good example similar to this problem is shown in the problem 2013 AB6. The grading standard usually shows a variation of the form $y=mx+b$, but other equivalent forms of an equation of a line are acceptable on the AP Exam for earning a point.

(2) We look for horizontal tangents (along with points where a derivative does not exist) to locate possible max and min values of a function. A derivative expressed as a rational function (or in "fractional form") provides good practice for both of these possibilities: let either the numerator or the denominator equal zero. In general, we find a vertical tangent line where the denominator equals 0. However, this is (sometimes?) said to exist only if the difference quotient has a limit of either $\pm\infty$. In other words, we must have $\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}=$ either $+\infty$ or $-\infty$, but not different infinities from left and right. To pursue this idea, examine the derivative of the function $f(x)=x^{2/3}$ at $x=0$ and decide for yourself if there is a vertical tangent line where $x=0$, despite the fact that $f'(x)=\frac{2}{3x^{1/3}}$ and the denominator of $f'(x)$ is 0 at $x=0$.

(3) The quotient rule without the denominator (squared) is not considered a quotient rule in the current grading of the AP Exam and earns zero points. Missing parentheses are treated more leniently. Regarding the rest of the quotient rule, students should be told and trained to know that **a quotient rule setup without parentheses in the numerator is probably doomed to failure**. A quotient rule involving an implicit differentiation requires practice above and beyond standard quotient rule practice. A nice example is found in 2004 AB4.