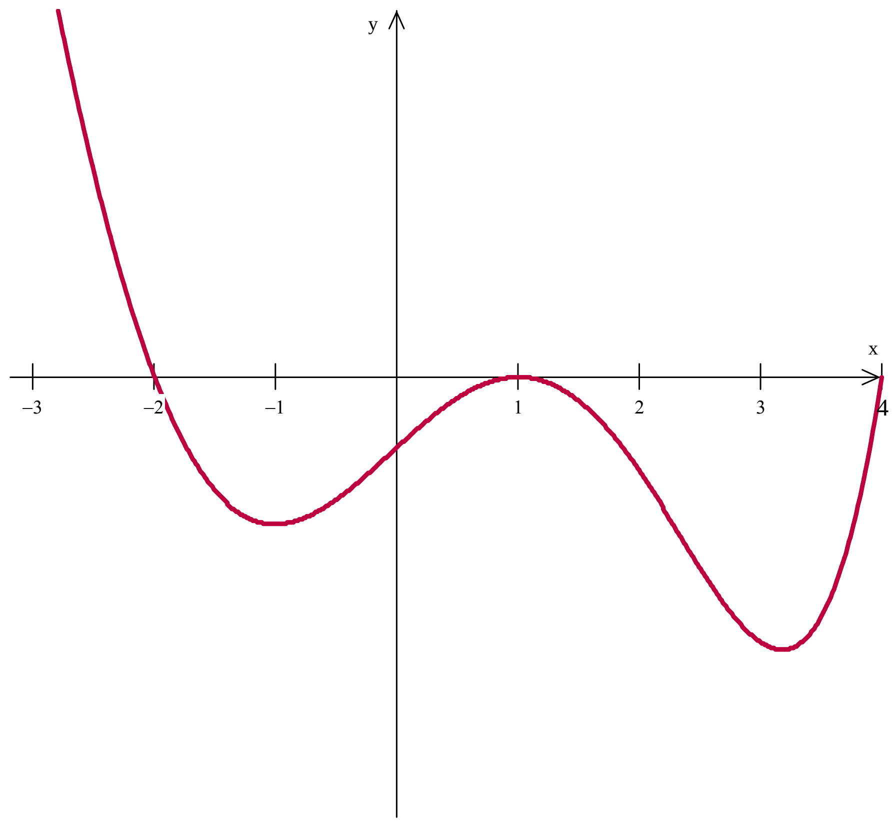
Problem Overview:

Students are given a graph of f' on $[-3, 4]$ and told that f is a twice differentiable function. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis, and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12 respectively. The graph of f' is at right.



Part a:

Students are asked to find all x-coordinates at which f has a relative maximum and give a reason for their answer.

Part b:

Students are asked to find the open intervals on $-3 < x < 4$ where the graph of f is both concave down and decreasing. Students need to provide a reason for their answer.

Part c:

Students are asked to find the x-coordinates of all points of inflection on the graph of f, again giving a reason for their answer.

Part d:

Students are given that $f(1) = 3$ and asked for an expression for f involving an integral as well as the values of $f(4)$ and $f(-2)$.

Part a:

For one point, students had to identify –2 as a potential answer. This could be done in the context of considering any subset of $\{-3, -2, 1. 4\}$. For the second point, we are done with reading for a "potential" answer and had to see –2 identified as the answer along with a reason. Acceptable reasons involved describing the derivative of f or the slopes of f as changing from positive to negative. Using phrases such as "the graph of f changes from increasing to decreasing" or "f' changes signs" were not acceptable reasons. Some students went too far and described the slope of f'.

Part b:

For the first point, both correct intervals had to be declared. Use of any subset of our intervals lost this point but made the student eligible for the second, reason, point. An incorrect interval outside of the correct intervals lost the first point and made the student ineligible for the second point. A common error was the use of the interval (–2, 1) rather than (–2, –1). Because of the graph, it was difficult to determine if this was a copy error or a mathematical error, but readers were told to treat this as a mathematical error and award no points in part (b). The second point was for the reason which had to communicate the fact that $f' < 0$ and $f'' < 0$ or state this in words such as "the graph of f' is below the x-axis and f' has a negative slope." Some students went too far and talked about the slope of f' decreasing, which is the same as saying that $f''' < 0$.

Part c:

A point of inflection is located where f'' changes sign, either from positive to negative or vice versa. This concept needs to be well established for students to appropriately search for values of x. Along with this concept, students should be well versed in ways to determine the signs of f''. Certainly this can come from information given for specific closed form expressions of f or f' or f''; but on the AP test that type of information is not often given. Therefore, determining signs of f'' should be practiced in the context of information given in different formats, such as the graph of f' as in this problem or in a table describing signs in an interval.

Part d:

The first, conceptual point was for an integrand of $f'(t)$ or $f'(x)$ appearing in a definite integral. Correct limits on this integral were part of the second point for an expression for f. Correct expressions for f could involve misuse of the dummy variable such as $3+\int_1^x f'(x)\,dx$ and be awarded the second point. Use of $f(1)$ rather than 3 was acceptable since that equivalence was given in the problem. A missing differential was ignored unless of the form $\int_1^x f'(t) + 3$. In that case, if both values for $f(4)$ and $f(-2)$ were correct, students were awarded one of the first two points and the third point for these values. If either value was incorrect in the presence of $\int_1^x f'(t) + 3$, students received zero points in part (d).

Personal note as a reader: the fact that 9 and 12 were given as areas in this problem and that –9 and 12 were answers in part (d) made grading this problem a bit tricky at times.

Observations and recommendations for teachers:

(1) The search for a maximum or minimum value of a function should be guided by a search for a change in signs of the derivative of the function, NOT solely by where $f' = 0$, a leap of faith that takes students out of the possibility (not in this problem) that f' does not exist. In the case of part (a), the graph provided sign information about the derivative of the function *f*. This type of setup should be practiced, because it was clear that some students confused this with the max or min on the graph of f'. On past tests, information about f' has been given in a graph (and also in a table, as in 2014 AB5). The reason must specify whether the sign of f' is changing from positive to negative or negative to positive in order to justify a max or min.

(2) In part (b), eligibility for a reason point requires that student answers regarding intervals remain within the correct intervals. A beautifully worded reason cannot be used to justify an incorrect answer.

Note that just knowing that $f' < 0$ and $f'' < 0$ earned the reason point for eligible students. The conditions "concave down and decreasing" should inspire students to state this fact, which was awarded the reason point. Knowing this fact might have helped some students avoid the error of declaring the interval (–2, 1) rather than (–2, –1). Knowing that the slopes of f' give us information about f'' is important also.

(3) At this point, it is also worthy to note (oops!!! ……… I just did it!!!) that use of the word "it" is NOT a good thing in justifying answers, on a math test or anywhere else. If ambiguity rules, tests will be graded harshly; and other documents can be deliberately misread by those with ulterior motives.

(4) Student work in part (c) made it clear that finding appropriate information about f'' was an adventure for many students. The core concept of a change in sign for f'' needs to be examined, known by students, and studied in a variety of contexts. Information about f'' can be gleaned from information about f' or from explicit expressions or from information about signs and values given in a table. All of these setups should be practiced in the classroom.

(5) Part (d) should remind us that students are not always doing analytic work calculating antiderivatives or derivatives in a problem that is not calculator-active. The setup is an important skill being tested, even though sometimes a missing or misused differential is not penalized. However, if any ambiguity results because of a missing differential, students will be penalized. It's best in the classroom to require the differential and train students accordingly. A way to emphasize this is in contexts such as using a definite integral to calculate area, volume or distance. The product of the integrand and the differential can be analyzed dimensionally, as in, for example, velocity × time = $v(t)\frac{\text{m}}{\text{sec}} \times dt\,\text{sec} = \text{meters}$. I have found students less likely to forget a differential if, in my class, early calculations of definite integrals involve such units.

(6) A good exercise is to find specific functions for *f*, f', and f''. This can be accomplished by looking at f' piecewise. For the portion of f', $x \geq 1$ note that the zeros are 1 and 4. By making the zero at 1 have multiplicity 2, a polynomial for f' can be constructed that satisfies the condition that $f''(3) = 0$. If I tell you too much more, I would take all the fun out of this.