2015 AB/BC4 Marshall Ransom, Georgia Southern University

Problem Overview:

Students were given the differential equation $\frac{dy}{dx} = 2x - y$.

Part a:

Students were asked to sketch a slope field at six given points on axes provided: on the y-axis at $y = -1,\ 1$ and 2 and for $x = 1$ at $y = -1,\ 1$ and 2.

Part b:

Students had to find $\frac{d^2y}{dx^2}$ in terms of x and y; then giving a reason for the answer, determine the concavity of all solutions for the differential equation in Quadrant II.

Part c:

Assuming $y = f(x)$ to be a particular solution to the differential equation with the initial condition $f(2) = 3$, determine whether f has a relative max, min or neither at the point where $x = 2$, and justify this.

Part d:

Assuming $y = mx + b$ is a solution to the differential equation, calculate the values of m and b.

Part a:

The slope field had to show the two negative slopes at (0, 2) and (0, 1) and one positive slope at (0, –1) for the first point. The second point was earned for a horizontal segment at (1, 2) and two positive slopes at the points (1, 1) and (1, –1). Points are awarded if the relative steepness is indicated correctly, or at least no worse than by parallel segments.

Part b:

$\frac{d^2y}{dx^2}$ had to be expressed in terms of x and y. Thus $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$ was not sufficient to earn the first point. However, a student with this expression could correctly refer to the signs of $x,\ y,$ and $\frac{dy}{dx}$ and declare "concave up" and be awarded the second point. The more revealing $\frac{d^2y}{dx^2} = 2 - (2x - y)$ could be verified as positive since $y > 0$ and $x < 0$ in Quadrant II, and that would justify an answer of "concave up." Mishandling of the negative sign as in $\frac{d^2y}{dx^2} = 2 - (2x - y) = 2 - 2x - y$ earned the first point but not the second. Starting with $\frac{d^2y}{dx^2} = 2 - 2x - y$ did not earn the first point, but this student was eligible for the second point if subsequent work showed use of the actual second derivative.

Part c:

For the first point, students had to be considering $\left.\frac{dy}{dx}\right|_{(2,3)}$ in such a manner as $f'(2) = 1$ or showing even more arithmetic work that connected (2, 3) or the value of 1 to f' or $\frac{dy}{dx}$; $2(2) - 3$ by itself was considered such a connection and earned the first point. Saying "$2(2) - 3 \neq 0$ so neither" earned both points. The second point required "neither" or "no" along with work establishing $f' \neq 0$.

Part d:

The philosophy in scoring for the first two points required a connection between the given differential equation and the slope m of the linear solution. This connection could be done minimally by merely asserting that $m = 2x - y$. The third point was for using that connection successfully to get the values of m and b. A few students chose to solve this differential equation either by using an integrating factor (not a topic for AP Calculus) or by using the substitution $u = 2x - y$ which results in a version of the given information that allows a solution for u by separation of variables. These approaches were rarely seen.

Observations and recommendations for teachers:

(1) A slope field showing very few points is not very revealing, making it probably best to practice this with far more than the six points given on this exam. A short line segment is preferred. Relative steepness of the segments is more important than trying to get a good approximation of the actual slope unless that number is very large or very small. Sometimes, rather than asking students to sketch, a more detailed slope field is given as in 2014 AB6. Another example of a slope field to be sketched is seen in 2006 AB5.

(2) An interesting example of calculating a second derivative, given the first, is found in 2004 AB4. Students need to practice calculating a derivative of a derivative, not merely starting with a polynomial, rational, or trig function, but starting with a given derivative. The second derivative reveals information about concavity, among other things. Verifying the sign of a somewhat awkward second derivative involves algebra and arithmetic skills. Note that students in part (b) had to appeal to signs of x and y in Quadrant II.

(3) It is not wise to send beginning calculus students into the larger world of mathematics believing that extrema only occur where a derivative is equal to zero. In this problem, that was the only possibility, but the teaching of and about extrema should be considered also at points where a derivative does not exist.

(4) This problem is somewhat unique on the AP exam, not because there are questions about a differential equation, but because there is no request for and no need for a solution to that equation. Practice answering questions about a function when information can be determined from a differential equation is essential. Another example, although a solution was ultimately required, is 2012 AB5.