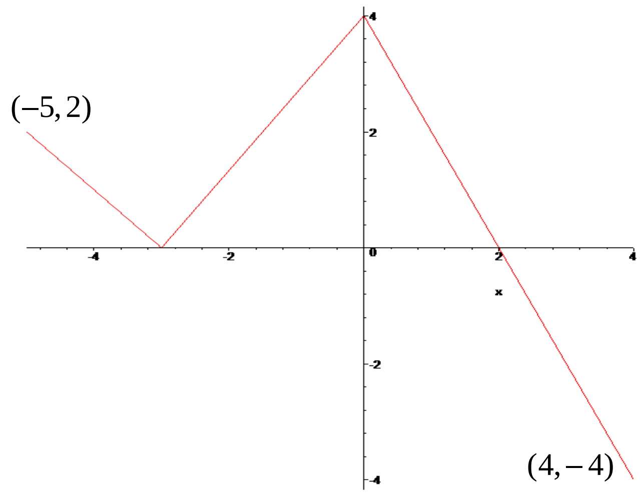
Problem Overview:



Students were given the graph of a function f shown at right consisting of three line segments. The function $g(x)=\int_{-3}^{x} f(t)\,dt$ was defined.

(Note: the graph on the test only labeled 1 on the x-axis and 1 on the y-axis, as well as the two points shown).

Part (a) says "Find $g(3)$."

Part (b) asked for the open intervals contained in $-5<x<4$ where g is both increasing and concave down.

Part (c) defined the function $h(x)=\dfrac{g(x)}{5x}$ and asked the students to find the value of $h'(3)$.

Part (d) defined the function $p(x)=f\left(x^2-x\right)$, and students were asked to find the slope of the line tangent to the graph of p at the point where $x=1$.

Part a:

There was only one point awarded for evaluating the integral $g(3)=\int_{-3}^{3} f(t)\,dt$. Using areas of three triangles, the first two of which are obviously $\frac{1}{2}(3)(4)=6$ and $\frac{1}{2}(4)(2)=4$, gives a final result of 9. Some students calculated the area of the entire triangle from $-3\le x\le 2$. The twist for many students was that the area of the third triangle (moving from left to right, which is under the x-axis) needs to be negative in calculating the definite integral.

Part b:

If both intervals were given, the first point was earned. The reasoning point required the equivalent of these two statements: $f>0$ and f is decreasing. Students could say "the graph" which was interpreted as referring to the only graph given in the problem, when trying to say that f was positive or decreasing. References to $g'>0$ and $g''<0$ had to be given in the presence of some reference to the fact that $g'=f$ or $g''=f'$ in order to earn the second, reasoning, point (which could be shown in part (a)). If only one of the intervals was given (and there were no incorrect intervals included) students were eligible for the reasoning point. Again, statements such as "the slope is negative," possibly referring to the function f, were not sufficient without saying, "the slope of f" or "the slope on the graph."

Part c:

Some acceptable answers (for the first two points) are shown below:

$$\frac{5xg'(x)-g(x)5}{(5x)^2},\quad \frac{xg'(x)-g(x)}{5x^2},\quad \frac{5xf(x)-5\int_{-3}^{x} f(t)dt}{(5x)^2} \quad \text{or} \quad \frac{5xg'(3)-g(3)5}{(5x)^2}$$

An answer worth 1 of the first 2 points might have left off parentheses as in $\frac{5xg'(x)-g(x)5}{5x^2}$. With this answer, the student was eligible for the answer point if the answer was our correct answer, as though the parentheses were actually there in the denominator. But the student never regained the second point for the derivative with this "presentation" error. The only exception to needing to see "our correct answer" was if the student imported an incorrect value for $g(3)$ from part (a).

Part d:

The key to the 2 derivative points here is the chain rule. Varieties of "almost" chain rule were accepted for 1 of these 2 points, such as $f'\left(x^2-x\right)2x$ or $f'\left(x^2-x\right)(2x+1)$. These were not eligible for the answer point. $f'\left(x^2+x\right)(2x+1)$ was worth 1 of the 2 derivative points and was interpreted as a "copy error" in which case readers have taken one point off for the copy error and this student is eligible for the answer point. A student had to earn one of the two derivative points to be eligible for the answer point. The answer required slope only, but a correct tangent line equation was also awarded an (eligible) answer point.

Observations and recommendations for teachers:

(1) For part (a), students needed to use areas to calculate a definite integral. This is a classic connection of areas to definite integrals of functions of one variable. A variety of practice is encouraged using regions both above and below the x-axis as well as semi-circular or quarter-circular regions. Examples abound in recent AP Calculus Exams, such as AB/BC3 on the 2012 exam.

(2) Students are asked about properties of a function g and a graph, where the function g is defined as an integral of a function f represented by a graph. The properties of g depend on the values of f, which as derivatives of g, determine regions of increase and decrease. The slopes of f represent the second derivative of such a g and thus pertain to concavity. Zeros of the function f as seen on the graph are potential max/min depending on the change in sign of f. Again, past AP Calculus Exam questions provide examples of questions that easily arise from such a graph.

(3) The quotient rule, with needed parentheses, should be practiced.

(4) A variety of chain rule situations should be practiced in class. Some suggestions include: $\frac{d}{dx}f\left(u(x)\right)$, $\frac{d}{dx}\sin\left(5\pi x^2\right)$, $\frac{d}{dx}e^{10\tan(x)}$, $\frac{d}{dx}e^{f(x)}$ or multiple applications of the chain rule such as

$\frac{d}{dx}\sin\left(5\tan\left(x^2\right)\right)$ or $\frac{d}{dx}e^{f(g(x))}$ or $\frac{d}{dx}\ln\left(f\left(g(x)\right)\right)$. On the exam, a chain rule error results in 0 points.