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Problem Overview:

Students were given the table below, describing behavior and a few values of f and g (which are twice differentiable functions) and values of derivatives at $x = -2,\ -1,\ 1\ \&\ 3$.

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

In part (a) students were asked to find the x-coordinate of each relative minimum of f on interval [–2, 3]. The answer(s) had to be justified. In part (b) students were asked to explain why there must be a value c such that $-1 < c < 1$ and $f''(c) = 0$. Part (c) defined a new function $h(x) = \ln(f(x))$. Students were asked to show computations that lead to the value of $h'(3)$. Part (d) required an evaluation of $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

Part a:

There was only one point awarded for stating both that $x = 1$ is the x-coordinate and why. This is because this is the only critical point at which $f'(x)$ changes from negative to positive in sign. The point was not awarded for a justification using vague language such as "the derivative" (the derivative of what?), "the slope" (the slope of what?), or " $f'(x)$ changes sign" (must specify change from negative to positive).

Part b:

This value of c exists because $f'(x)$ is differentiable, hence also continuous and meets the criteria of the Mean Value Theorem (MVT). The requisite slope of a secant line (that needs to be the same as the slope of a tangent line, as in the value of $f''(c)$) is $\dfrac{f'(1) - f'(-1)}{1-(-1)} = \dfrac{0-0}{2} = 0$. The first point was for showing an attempt at working with $f'(1) - f'(-1)$. The second point was awarded for citing the MVT or Rolle's Theorem (which is valid in this case because at the endpoints of the interval $f'(1) = f'(-1) = 0$) or the continuity of $f'(x)$ (which continuity is based upon the differentiability of $f'(x)$, but merely citing this differentiability was not sufficient to earn the point). For example, a "minimally" correct answer for both points was " $f'(1) = f'(-1)$ so by Rolle's theorem there must be such a c." Attempts were made by students to invoke the Extreme Value Theorem, not very well applicable in this situation.

Part c:

The first two points were for calculating $h'(x)$. The students had to show **evidence of use of the chain rule**. The derivative could be expressed as $\frac{f'(x)}{f(x)}$ or students could have begun to use the value $x = 3$ in a manner such as $h'(x) = \frac{f'(3)}{f(3)}$, a bit sloppy, but indicating a correct attempt at the derivative of this log and the use of the chain rule. Some work had to be shown. Thus $\frac{f'(3)}{7}$ was acceptable but $h'(3) = \frac{1/2}{7}$ was not since there is no indication of how the values were obtained (this student was, however, awarded the answer point). For one out of the first two points, **readers had to see the chain rule** in combination with a clear attempt at the derivative of the natural log function such as $h'(x) = \frac{f(x)}{x} f'(x)$ or $\frac{1}{\ln(f(x))} f'(x)$. Such answers did receive one of the first two points but were not eligible for the third, answer point. The answer point was awarded only for the actual correct answer and not for using values related to incorrect work attempts at the first two points.

Part d:

The essence of this question is applying the Fundamental Theorem of Calculus (FTC) as in, simplistically, "the integral of the derivative of a function is that function." Students needed to have a correct anti-derivative and a correct use of limits to earn the first two points. Answers with an implicit u-substitution such as $\int_{-2}^{3} f'(u)\,du = f(u)\big|_{-2}^{3} = f(1) - f(-1)$ earned both the first two points and were eligible for the answer point. In order to be eligible for the answer point, students needed to earn at least one of the two FTC points. The use of u had to be explicitly connected to g as in $u = g(x)$ or implicitly connected as in $\int_{-1}^{1} f'(u)\,du$. Without this connection, students earned 0 out of the first two points and were ineligible for the answer point. Some students had $\int_{-2}^{3} f'(u)\,du = f(u)\big|_{-2}^{3} = f(3) - f(-2)$ and received no points for this work. But $u = g(x)$ and $\int_{-2}^{3} f'(u)\,du = f(u)\Big|_{-2}^{3} = f(-1) - f(1) = 6$ earned one of the first two points and the answer point, despite the reversal. Thus, the answer point could be awarded for +6 in the case of a reversal that earned one of the first two FTC points.

Observations and recommendations for teachers:

(1) For part (a), students need to know that a **relative** extremum exists when the derivative changes sign. But to specify a max or min, students do need to specify that the sign change is from positive to negative or vice versa. Also, the change is in the sign of the derivative of the function in question. Without reference to that specific derivative, the student's argument is insufficient. References to slope, as a synonym for "derivative," can work as long as it is clear that the slope refers to the function in question. For example, "the slope changes from negative to positive" is not sufficient to justify a relative minimum because we need to know what slope this is, as in "the slope of f."

(2) The Mean Value Theorem hypotheses are (i) continuity on a closed interval and (ii) differentiability on an open interval. Students do not need to specify these. Mention of the MVT, if it applies as in part (b), is sufficient. But some work needs to be shown in calculating the slope of the secant line. The idea of continuity inferred from the differentiability of the function in question (in the case of part (b) that is $f'(x)$) must be mentioned in the absence of reference to the MVT in order to make a solid argument and justification. It is also true that if the function in question has identical values at the endpoints, then Rolle's Theorem can be invoked.

(3) Students need to know the derivative of a natural log function (as quite simply "one over the argument of the log function"). Of greater importance in the scoring of an AP exam question is correct use of the chain rule. When the chain rule is missing, students will typically receive 0 of the points allotted for derivative work and will not be eligible for an answer point. It is not unusual on an AP exam question involving analysis of a function to create a new composite function, requiring a chain rule in calculation of the derivative. A variety of chain rule situations should be practiced in class. Some suggestions include:

$\frac{d}{dx} f\left(u(x)\right),\ \frac{d}{dx}\sin\left(5\pi x^2\right),\ \frac{d}{dx}e^{10\tan(x)},\ \frac{d}{dx}e^{f(x)}$ or multiple applications of the chain rule such as

$\frac{d}{dx}\sin\left(5\tan\left(x^2\right)\right)$ or $\frac{d}{dx}e^{f(g(x))}$ or $\frac{d}{dx}\ln\left(f\left(g(x)\right)\right)$.

(4) For a number of years, the FTC has appeared on the AP exam in a variety of ways. In the case of AB5 part (d) this appears and is best handled using a u-substitution. While some simple u-substitutions do not require work to be shown such as $\int \cos(5t)\,dt = \frac{\sin(5t)}{5} + C$, part (d) did require some indication of knowledge of the function in the composite. Somehow, either stating that $u = g(x)$ or with a good change of the limits on the integral, a connection had to be made so that the antiderivative made sense and readers could see that the student knew where it came from and/or where it was going.